THEORY OF THE SIEGEL MODULAR VARIETY

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ABSTRACT. In this paper, we discuss the theory of the Siegel modular variety in the aspects of arithmetic and geometry. This article covers the theory of Siegel modular forms, the Hecke theory, a lifting of elliptic cusp forms, geometric properties of the Siegel modular variety, (hypothetical) motives attached to Siegel modular forms.

To the memory of my mother

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1. Introduction

For a given fixed positive integer g, we let

$$\mathbb{H}_g = \{\, \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \,\}$$

be the Siegel upper half plane of degree g and let

$$Sp(g,\mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \,\}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transposed matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Then $Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$. Let

$$\Gamma_g = Sp(g,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}) \,\Big|\, A,B,C,D \text{ integral} \right\}$$

be the Siegel modular group of degree g. This group acts on $\mathbb{H}_g$ properly discontinuously. C. L. Siegel investigated the geometry of $\mathbb{H}_g$ and automorphic forms on $\mathbb{H}_g$ systematically. Siegel [131] found a fundamental domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$ and described it explicitly. Moreover he calculated the volume of $\mathcal{F}_g$. We also refer to [65], [92], [131] for some details on $\mathcal{F}_g$. Siegel's fundamental domain is now called the Siegel modular variety and is usually denoted by $\mathcal{A}_g$. In fact, $\mathcal{A}_g$ is one of the important arithmetic varieties in the sense that it is regarded as the moduli of principally polarized abelian varieties of dimension g. Suggested by Siegel, I. Satake [117] found a canonical compactification, now called the Satake compactification of $\mathcal{A}_g$. Thereafter W. Baily [6] proved that the Satake compactification of $\mathcal{A}_g$ is a normal projective variety. This work was generalized to bounded symmetric domains by W. Baily and A. Borel [7] around the 1960s. Some years later a theory of smooth compactification of bounded symmetric domains was developed by Mumford school [5]. G. Faltings and C.-L. Chai [30] investigated the moduli of abelian varieties over the integers and could give the analogue of the Eichler-Shimura theorem that expresses Siegel modular forms in terms of the cohomology of local systems on $\mathcal{A}_g$. I want to emphasize that Siegel modular forms play an important role in the theory of the arithmetic and the geometry of the Siegel modular variety $\mathcal{A}_g$.

The aim of this paper is to discuss a theory of the Siegel modular variety in the aspects of arithmetic and geometry. Unfortunately two important subjects, which are the theory of harmonic analysis on the Siegel modular variety, and the Galois representations associated to Siegel modular forms are not covered in this article. These two topics shall be discussed in the near future in the separate papers. This article is organized as follows. In Section 2, we review the results of Siegel and Maass on invariant metrics and their Laplacians on $\mathbb{H}_g$. In Section 3, we investigate differential operators on $\mathbb{H}_g$ invariant under the action

(1.1). In Section 4, we review Siegel's fundamental domain $\mathcal{F}_g$ and expound the spectral theory of the abelian variety A_Ω associated to an element Ω of $\mathcal{F}_g$. In Section 5, we review some properties of vector valued Siegel modular forms, and also discuss construction of Siegel modular forms and singular modular forms. In Section 6, we review the structure of the Hecke algebra of the group $GSp(g,\mathbb{Q})$ of symplectic similitudes and investigate the action of the Hecke algebra on Siegel modular forms. In Section 7, we briefly illustrate the basic notion of Jacobi forms which are needed in the next section. We also give a short historical survey on the theory of Jacobi forms. In Section 8, we deal with a lifting of elliptic cusp forms to Siegel modular forms and give some recent results on the lifts obtained by some people. A lifting of modular forms plays an important role arithmetically and geometrically. One of the interesting lifts is the so-called Duke-Imamoḡlu-Ikeda lift. We discuss this lift in some detail. In Section 9, we give a short survey of toroidal compactifications of the Siegel modular variety $\mathcal{A}_g$ and illustrate a relationship between Siegel modular forms and holomorphic differential forms on $\mathcal{A}_g$. Siegel modular forms related to holomorphic differential forms on $\mathcal{A}_g$ play an important role in studying the geometry of $\mathcal{A}_g$. In Section 10, We investigate the geometry of subvarieties of the Siegel modular variety. Recently Grushevsky and Lehavi [45] announced that they proved that the Siegel modular variety $\mathcal{A}_6$ of genus 6 is of general type after constructing a series of new effective geometric divisors on $\mathcal{A}_g$. Before 2005 it had been known that $\mathcal{A}_g$ is of general type for $g \geq 7$. In fact, in 1983 Mumford [102] proved that $\mathcal{A}_g$ is of general type for $g \geq 7$. Nearly past twenty years nobody had known whether $\mathcal{A}_6$ is of general type or not. In Section 11, we formulate the proportionality theorem for an automorphic vector bundle on the Siegel modular variety following the work of Mumford (cf. [101]). In Section 12, we explain roughly Yoshida's interesting results about the fundamental periods of a motive attached to a Siegel modular form. These results are closely related to Deligne's conjecture about critical values of an L-function of a motive and the (pure or mixed) Hodge theory.

Finally I would like to give my hearty thanks to Hiroyuki Yoshida for explaining his important work kindly and sending two references [162, 163] to me.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transposed matrix of M. I_n denotes the identity matrix of degree n. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a number field F, we denote by $\mathbb{A}_F$ the ring of adeles of F. If $F = \mathbb{Q}$, the subscript will be omitted. We denote by $\mathbb{A}_{F,f}$ and $\mathbb{A}_f$ the finite part of $\mathbb{A}_F$ and $\mathbb{A}$ respectively. By $\overline{\mathbb{Q}}$ we mean the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

2. **Invariant Metrics and Laplacians on Siegel Space**

For $\Omega = (\omega_{ij}) \in \mathbb{H}_g$, we write $\Omega = X + iY$ with $X = (x_{ij}),\ Y = (y_{ij})$ real and $d\Omega = (d\omega_{ij})$. We also put

$$\frac{\partial}{\partial\Omega} = \left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\omega_{ij}}\right) \qquad \text{and} \qquad \frac{\partial}{\partial\overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\overline{\omega}_{ij}}\right).$$

C. L. Siegel [131] introduced the symplectic metric ds^2 on $\mathbb{H}_g$ invariant under the action (1.1) of $Sp(g,\mathbb{R})$ given by

$$ds^2 = \sigma(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}) \tag{2.1}$$

and H. Maass [91] proved that its Laplacian is given by

$$\Delta = 4\,\sigma\left(Y\ {}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right). \tag{2.2}$$

And

$$dv_g(\Omega) = (\det Y)^{-(g+1)} \prod_{1\le i\le j\le g} dx_{ij} \prod_{1\le i\le j\le g} dy_{ij} \tag{2.3}$$

is a $Sp(g,\mathbb{R})$-invariant volume element on $\mathbb{H}_g$ (cf. [133], p. 130).

Theorem 2.1. (**Siegel** [131]). *(1) There exists exactly one geodesic joining two arbitrary points $\Omega_0,\ \Omega_1$ in $\mathbb{H}_g$. Let $R(\Omega_0,\Omega_1)$ be the cross-ratio defined by*

$$R(\Omega_0,\Omega_1) = (\Omega_0-\Omega_1)(\Omega_0-\overline{\Omega}_1)^{-1}(\overline{\Omega}_0-\overline{\Omega}_1)(\overline{\Omega}_0-\Omega_1)^{-1}. \tag{2.4}$$

For brevity, we put $R_* = R(\Omega_0,\Omega_1)$. Then the symplectic length $\rho(\Omega_0,\Omega_1)$ of the geodesic joining Ω_0 and Ω_1 is given by

$$\rho(\Omega_0,\Omega_1)^2 = \sigma\left(\left(\log\frac{1+R_*^{\frac12}}{1-R_*^{\frac12}}\right)^2\right), \tag{2.5}$$

where

$$\left(\log\frac{1+R_*^{\frac12}}{1-R_*^{\frac12}}\right)^2 = 4\,R_*\left(\sum_{k=0}^{\infty}\frac{R_*^k}{2k+1}\right)^2.$$

(2) For $M \in Sp(g,\mathbb{R})$, we set

$$\tilde{\Omega}_0 = M\cdot\Omega_0 \quad \text{and} \quad \tilde{\Omega}_1 = M\cdot\Omega_1.$$

Then $R(\Omega_1,\Omega_0)$ and $R(\tilde{\Omega}_1,\tilde{\Omega}_0)$ have the same eigenvalues.
(3) All geodesics are symplectic images of the special geodesics

$$\alpha(t) = i\ diag(a_1^t, a_2^t, \cdots, a_g^t), \tag{2.6}$$

where $a_1, a_2, \cdots, a_g$ are arbitrary positive real numbers satisfying the condition

$$\sum_{k=1}^{g}(\log a_k)^2 = 1.$$

The proof of the above theorem can be found in [131], pp. 289-293.

Let

$$\mathbb{D}_g = \Big\{ W \in \mathbb{C}^{(g,g)} \mid W = {}^tW,\ I_g - W\overline{W} > 0 \Big\}$$

be the generalized unit disk of degree g. The Cayley transform $\Psi : \mathbb{D}_g \longrightarrow \mathbb{H}_g$ defined by

$$\Psi(W) = i\,(I_g + W)(I_g - W)^{-1}, \quad W \in \mathbb{D}_g \tag{2.7}$$

is a biholomorphic mapping of $\mathbb{D}_g$ onto $\mathbb{H}_g$ which gives the bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$ (cf. [131]). A. Korányi and J. Wolf [81] gave a realization of a bounded symmetric domain as a Siegel domain of the third kind investigating a generalized Cayley transform of a bounded symmetric domain that generalizes the Cayley transform Ψ of $\mathbb{D}_g$.

Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_g & I_g \\ iI_g & -iI_g \end{pmatrix} \tag{2.8}$$

be the $2g \times 2g$ matrix represented by Ψ. Then

$$T^{-1} Sp(g,\mathbb{R})\, T = \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \,\Big|\, {}^tP\overline{P} - {}^t\overline{Q}Q = I_g,\ {}^tP\overline{Q} = {}^t\overline{Q}P \right\}. \tag{2.9}$$

Indeed, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$, then

$$T^{-1} M T = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \tag{2.10}$$

where

$$P = \frac{1}{2} \Big\{ (A + D) + i\,(B - C) \Big\} \tag{2.11}$$

and

$$Q = \frac{1}{2} \Big\{ (A - D) - i\,(B + C) \Big\}. \tag{2.12}$$

For brevity, we set

$$G_* = T^{-1} Sp(g,\mathbb{R}) T.$$

Then G_* is a subgroup of $SU(g,g)$, where

$$SU(g,g) = \Big\{ h \in \mathbb{C}^{(g,g)} \,\big|\, {}^th I_{g,g} \overline{h} = I_{g,g} \Big\}, \quad I_{g,g} = \begin{pmatrix} I_g & 0 \\ 0 & -I_g \end{pmatrix}.$$

In the case $g = 1$, we observe that

$$T^{-1} Sp(1,\mathbb{R}) T = T^{-1} SL_2(\mathbb{R}) T = SU(1,1).$$

If $g > 1$, then G_* is a *proper* subgroup of $SU(g,g)$. In fact, since ${}^tT J_g T = -\,i\, J_g$, we get

$$G_* = \Big\{ h \in SU(g,g) \,\big|\, {}^th J_g h = J_g \Big\} = SU(g,g) \cap Sp(g,\mathbb{C}), \tag{2.13}$$

where

$$Sp(g,\mathbb{C}) = \Big\{ \alpha \in \mathbb{C}^{(2g,2g)} \ \big|\ {}^t\alpha\, J_g\, \alpha = J_g \Big\}.$$

Let

$$P^+ = \left\{ \begin{pmatrix} I_g & Z \\ 0 & I_g \end{pmatrix} \,\Big|\, Z = {}^tZ \in \mathbb{C}^{(g,g)} \right\}$$

be the P^+-part of the complexification of $G_* \subset SU(g,g)$. We note that the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}$ in G_* is

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} = \begin{pmatrix} I_g & Q\overline{P}^{-1} \\ 0 & I_g \end{pmatrix} \begin{pmatrix} P - Q\overline{P}^{-1}\overline{Q} & 0 \\ 0 & \overline{P} \end{pmatrix} \begin{pmatrix} I_g & 0 \\ \overline{P}^{-1}\overline{Q} & I_g \end{pmatrix}.$$

For more detail, we refer to [75, p. 155]. Thus the P^+-component of the following element

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot \begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, \quad W \in \mathbb{D}_g$$

of the complexification of G_*^J is given by

$$\begin{pmatrix} I_g & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_g \end{pmatrix}. \tag{2.14}$$

We note that $Q\overline{P}^{-1} \in \mathbb{D}_g$. We get the Harish-Chandra embedding of $\mathbb{D}_g$ into P^+ (cf. [75, p. 155] or [120, pp. 58-59]). Therefore we see that G_* acts on $\mathbb{D}_g$ transitively by

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot W = (PW+Q)(\overline{Q}W+\overline{P})^{-1}, \quad \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ W \in \mathbb{D}_g. \tag{2.15}$$

The isotropy subgroup K_* of G_* at the origin o is given by

$$K_* = \left\{ \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix} \,\Big|\, P \in U(g) \right\}.$$

Thus G_*/K_* is biholomorphic to $\mathbb{D}_g$. It is known that the action (1.1) is compatible with the action (2.15) via the Cayley transform Ψ (cf. (2.7)). In other words, if $M \in Sp(g,\mathbb{R})$ and $W \in \mathbb{D}_g$, then

$$M \cdot \Psi(W) = \Psi(M_* \cdot W), \tag{2.16}$$

where $M_* = T^{-1}MT \in G_*$.

For $W = (w_{ij}) \in \mathbb{D}_g$, we write $dW = (dw_{ij})$ and $d\overline{W} = (d\overline{w}_{ij})$. We put

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial w_{ij}} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \overline{w}_{ij}} \right).$$

Using the Cayley transform $\Psi : \mathbb{D}_g \longrightarrow \mathbb{H}_g$, Siegel showed (cf. [131]) that

$$ds_*^2 = 4\sigma\Big((I_g - W\overline{W})^{-1} dW\, (I_g - \overline{W}W)^{-1} d\overline{W} \Big) \tag{2.17}$$

is a G_*-invariant Riemannian metric on $\mathbb{D}_g$ and Maass [91] showed that its Laplacian is given by

$$\Delta_* = \sigma\left((I_g - W\overline{W})\, {}^t\!\left((I_g - W\overline{W}) \frac{\partial}{\partial \overline{W}} \right) \frac{\partial}{\partial W} \right). \tag{2.18}$$

3. Invariant Differential Operators on Siegel Space

For brevity, we write $G = Sp(g,\mathbb{R})$. The isotropy subgroup K at iI_g for the action (1.1) is a maximal compact subgroup given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \;\Big|\; A\,{}^tA + B\,{}^tB = I_g,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(g,g)} \right\}.$$

Let $\mathfrak{k}$ be the Lie algebra of K. Then the Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \;\Big|\; X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(g,g)} \right\}.$$

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_g$ at iI_g. The adjoint representation of G on $\mathfrak{g}$ induces the action of K on $\mathfrak{p}$ given by

$$k \cdot Z = kZ\,{}^tk, \quad k \in K,\ Z \in \mathfrak{p}. \tag{3.1}$$

Let T_g be the vector space of $g \times g$ symmetric complex matrices. We let $\psi : \mathfrak{p} \longrightarrow T_g$ be the map defined by

$$\psi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \right) = X + i\,Y, \quad \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}. \tag{3.2}$$

We let $\delta : K \longrightarrow U(g)$ be the isomorphism defined by

$$\delta\left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right) = A + i\,B, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K, \tag{3.3}$$

where $U(g)$ denotes the unitary group of degree g. We identify $\mathfrak{p}$ (resp. K) with T_g (resp. $U(g)$) through the map Ψ (resp. δ). We consider the action of $U(g)$ on T_g defined by

$$h \cdot Z = hZ\,{}^th, \quad h \in U(g),\ Z \in T_g. \tag{3.4}$$

Then the adjoint action (3.1) of K on $\mathfrak{p}$ is compatible with the action (3.4) of $U(g)$ on T_g through the map ψ. Precisely for any $k \in K$ and $\omega \in \mathfrak{p}$, we get

$$\psi(k\,\omega\,{}^tk) = \delta(k)\,\psi(\omega)\,{}^t\delta(k). \tag{3.5}$$

The action (3.4) induces the action of $U(g)$ on the polynomial algebra $\mathrm{Pol}(T_g)$ and the symmetric algebra $S(T_g)$ respectively. We denote by $\mathrm{Pol}(T_g)^{U(g)}$ $\Big($resp. $S(T_g)^{U(g)}\Big)$ the subalgebra of $\mathrm{Pol}(T_g)$ $\Big($resp. $S(T_g)\Big)$ consisting of $U(g)$-invariants. The following inner product $(\ ,\ )$ on T_g defined by

$$(Z, W) = \mathrm{tr}\big(Z\,\overline{W}\,\big), \quad Z, W \in T_g$$

gives an isomorphism as vector spaces

$$T_g \cong T_g^*, \quad Z \mapsto f_Z, \quad Z \in T_g, \tag{3.6}$$

where T_g^* denotes the dual space of T_g and f_Z is the linear functional on T_g defined by

$$f_Z(W) = (W, Z), \quad W \in T_g.$$

It is known that there is a canonical linear bijection of $S(T_g)^{U(g)}$ onto the algebra $\mathbb{D}(\mathbb{H}_g)$ of differential operators on $\mathbb{H}_g$ invariant under the action (1.1) of G. Identifying T_g with T_g^* by the above isomorphism (3.6), we get a canonical linear bijection

$$\Phi : \mathrm{Pol}(T_g)^{U(g)} \longrightarrow \mathbb{D}(\mathbb{H}_g) \tag{3.7}$$

of $\mathrm{Pol}(T_g)^{U(g)}$ onto $\mathbb{D}(\mathbb{H}_g)$. The map Φ is described explicitly as follows. Similarly the action (3.1) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p})$ and $S(\mathfrak{p})$ respectively. Through the map ψ, the subalgebra $\mathrm{Pol}(\mathfrak{p})^K$ of $\mathrm{Pol}(\mathfrak{p})$ consisting of K-invariants is isomorphic to $\mathrm{Pol}(T_g)^{U(g)}$. We put $N = g(g+1)$. Let $\{\xi_\alpha \mid 1 \leq \alpha \leq N\}$ be a basis of $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Phi(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^N t_\alpha \xi_\alpha \right) K \right)\right]_{(t_\alpha)=0}, \tag{3.8}$$

where $f \in C^\infty(\mathbb{H}_g)$. We refer to [53, 54] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [46, 47], the algebra $\mathbb{D}(\mathbb{H}_g)$ is generated by g algebraically independent generators and is isomorphic to the commutative ring $\mathbb{C}[x_1, \cdots, x_g]$ with g indeterminates. We note that g is the real rank of G. Let $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$. It is known that $\mathbb{D}(\mathbb{H}_g)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ (cf. [129]).

Using a classical invariant theory (cf. [58, 147]), we can show that $\mathrm{Pol}(T_g)^{U(g)}$ is generated by the following algebraically independent polynomials

$$q_j(Z) = \mathrm{tr}\Big(\big(Z\overline{Z}\big)^j \Big), \quad j = 1, 2, \cdots, g. \tag{3.9}$$

For each j with $1 \leq j \leq g$, the image $\Phi(q_j)$ of q_j is an invariant differential operator on $\mathbb{H}_g$ of degree $2j$. The algebra $\mathbb{D}(\mathbb{H}_g)$ is generated by g algebraically independent generators $\Phi(q_1), \Phi(q_2), \cdots, \Phi(q_g)$. In particular,

$$\Phi(q_1) = c_1 \,\mathrm{tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) \quad \text{for some constant } c_1. \tag{3.10}$$

We observe that if we take $Z = X + i\,Y$ with real X, Y, then $q_1(Z) = q_1(X, Y) = \mathrm{tr}\big(X^2 + Y^2\big)$ and

$$q_2(Z) = q_2(X, Y) = \mathrm{tr}\Big(\big(X^2 + Y^2\big)^2 + 2\,X\big(XY - YX\big)Y \Big).$$

We propose the following problem.

Problem. Express the images $\Phi(q_j)$ explicitly for $j = 2, 3, \cdots, g$.

We hope that the images $\Phi(q_j)$ for $j = 2, 3, \cdots, g$ are expressed in the form of the *trace* as $\Phi(q_1)$.

Example 3.1. We consider the case $g = 1$. The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$q(z) = z\,\overline{z}, \quad z \in \mathbb{C}.$$

Using Formula (3.8), we get

$$\Phi(q) = 4\,y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\Phi(q)\big]$.

Example 3.2. We consider the case $g = 2$. The algebra $\mathrm{Pol}(T_2)^{U(2)}$ is generated by the polynomial

$$q_1(Z) = \sigma\big(Z\,\overline{Z}\,\big), \quad q_2(Z) = \sigma\Big(\big(Z\,\overline{Z}\big)^2\Big), \quad Z \in T_2.$$

Using Formula (3.8), we may express $\Phi(q_1)$ and $\Phi(q_2)$ explicitly. $\Phi(q_1)$ is expressed by Formula (3.10). The computation of $\Phi(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Phi(q_2)$ was essentially computed in [19], Proposition 6. Therefore $\mathbb{D}(\mathbb{H}_2) = \mathbb{C}\big[\Phi(q_1), \Phi(q_2)\big]$. The authors of [19] computed the center of $U(\mathfrak{g}_{\mathbb{C}})$.

4. **Siegel's Fundamental Domain**

We let

$$\mathcal{P}_g = \Big\{\, Y \in \mathbb{R}^{(g,g)} \mid Y = {}^tY > 0 \,\Big\}$$

be an open cone in $\mathbb{R}^N$ with $N = g(g+1)/2$. The general linear group $GL(g,\mathbb{R})$ acts on $\mathcal{P}_g$ transitively by

$$g \circ Y := gY\,{}^tg, \qquad g \in GL(g,\mathbb{R}),\ Y \in \mathcal{P}_g. \tag{4.1}$$

Thus $\mathcal{P}_g$ is a symmetric space diffeomorphic to $GL(g,\mathbb{R})/O(g)$.

The fundamental domain $\mathcal{R}_g$ for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$ which was found by H. Minkowski [97] is defined as a subset of $\mathcal{P}_g$ consisting of $Y = (y_{ij}) \in \mathcal{P}_g$ satisfying the following conditions (M.1)–(M.2) (cf. [65] p. 191 or [92] p. 123):

(M.1) $aY\,{}^ta \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^g$ in which $a_k, \cdots, a_g$ are relatively prime for $k = 1, 2, \cdots, g$.

(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \cdots, g-1$.

We say that a point of $\mathcal{R}_g$ is *Minkowski reduced* or simply *M-reduced.* $\mathcal{R}_g$ has the following properties (R1)–(R4):

(R1) For any $Y \in \mathcal{P}_g$, there exist a matrix $A \in GL(g,\mathbb{Z})$ and $R \in \mathcal{R}_g$ such that $Y = R[A]$ (cf. [65] p. 191 or [92] p. 139). That is,

$$GL(g,\mathbb{Z}) \circ \mathcal{R}_g = \mathcal{P}_g.$$

(R2) $\mathcal{R}_g$ is a convex cone through the origin bounded by a finite number of hyperplanes. $\mathcal{R}_g$ is closed in $\mathcal{P}_g$ (cf. [92] p. 139).

(R3) If Y and $Y[A]$ lie in $\mathcal{R}_g$ for $A \in GL(g,\mathbb{Z})$ with $A \neq \pm I_g$, then Y lies on the boundary $\partial\mathcal{R}_g$ of $\mathcal{R}_g$. Moreover $\mathcal{R}_g \cap (\mathcal{R}_g[A]) \neq \emptyset$ for only finitely many $A \in GL(g,\mathbb{Z})$ (cf. [92] p. 139).

(R4) If $Y = (y_{ij})$ is an element of $\mathcal{R}_g$, then

$$y_{11} \leq y_{22} \leq \cdots \leq y_{gg} \quad \text{and} \quad |y_{ij}| < \frac{1}{2} y_{ii} \quad \text{for } 1 \leq i < j \leq g.$$

We refer to [65] p. 192 or [92] pp. 123-124.

Remark. Grenier [43] found another fundamental domain for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$.

For $Y=(y_{ij})\in\mathcal{P}_g$, we put

$$dY=(dy_{ij})\qquad\text{and}\qquad \frac{\partial}{\partial Y}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial y_{ij}}\right).$$

Then we can see easily that

$$ds^2=\sigma((Y^{-1}dY)^2) \tag{4.2}$$

is a $GL(g,\mathbb{R})$-invariant Riemannian metric on $\mathcal{P}_g$ and its Laplacian is given by

$$\Delta=\sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right).$$

We also can see that

$$d\mu_g(Y)=(\det Y)^{-\frac{g+1}{2}}\prod_{i\le j}dy_{ij}$$

is a $GL(g,\mathbb{R})$-invariant volume element on $\mathcal{P}_g$. The metric ds^2 on $\mathcal{P}_g$ induces the metric $ds^2_{\mathcal{R}}$ on $\mathcal{R}_g$. Minkowski [97] calculated the volume of $\mathcal{R}_g$ for the volume element $[dY]:=\prod_{i\le j}dy_{ij}$ explicitly. Later Siegel computed the volume of $\mathcal{R}_g$ for the volume element $[dY]$ by a simple analytic method and generalized this case to the case of any algebraic number field.

Siegel [131] determined a fundamental domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$. We say that $\Omega=X+iY\in\mathbb{H}_g$ with X, Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\mathrm{Im}\,(\gamma\cdot\Omega))\le\det(\mathrm{Im}\,(\Omega))\qquad$ for all $\gamma\in\Gamma_g$;

(S.2) $Y=\mathrm{Im}\,\Omega$ is M-reduced, that is, $Y\in\mathcal{R}_g$;

(S.3) $|x_{ij}|\le\frac{1}{2}\quad$ for $1\le i,j\le g,$ where $X=(x_{ij})$.

$\mathcal{F}_g$ is defined as the set of all Siegel reduced points in $\mathbb{H}_g$. Using the highest point method, Siegel proved the following (F1)–(F3) (cf. [65] pp. 194-197 or [92] p. 169):

(F1) $\Gamma_g\cdot\mathcal{F}_g=\mathbb{H}_g$, i.e., $\mathbb{H}_g=\cup_{\gamma\in\Gamma_g}\gamma\cdot\mathcal{F}_g$.

(F2) $\mathcal{F}_g$ is closed in $\mathbb{H}_g$.

(F3) $\mathcal{F}_g$ is connected and the boundary of $\mathcal{F}_g$ consists of a finite number of hyperplanes.

The metric ds^2 given by (2.1) induces a metric $ds^2_{\mathcal{F}}$ on $\mathcal{F}_g$.

Siegel [131] computed the volume of $\mathcal{F}_g$

$$\mathrm{vol}\,(\mathcal{F}_g)=2\prod_{k=1}^{g}\pi^{-k}\,\Gamma(k)\,\zeta(2k), \tag{4.3}$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,

$$\mathrm{vol}\,(\mathcal{F}_1)=\frac{\pi}{3},\quad \mathrm{vol}\,(\mathcal{F}_2)=\frac{\pi^3}{270},\quad \mathrm{vol}\,(\mathcal{F}_3)=\frac{\pi^6}{127575},\quad \mathrm{vol}\,(\mathcal{F}_4)=\frac{\pi^{10}}{200930625}.$$

For a fixed element $\Omega \in \mathbb{H}_g$, we set

$$L_\Omega := \mathbb{Z}^g + \mathbb{Z}^g\Omega, \qquad \mathbb{Z}^g = \mathbb{Z}^{(1,g)}.$$

It follows from the positivity of $\mathrm{Im}\,\Omega$ that L_Ω is a lattice in $\mathbb{C}^g$. We see easily that if Ω is an element of $\mathbb{H}_g$, the period matrix $\Omega_* := (I_g, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2) :

(RC.1) $\quad \Omega_* J_g, {}^t\,\Omega_* = 0.$

(RC.2) $\quad -\frac{1}{i}\Omega_* J_g\, {}^t\overline{\Omega}_* > 0.$

Thus the complex torus $A_\Omega := \mathbb{C}^g/L_\Omega$ is an abelian variety.

We fix an element $\Omega = X + iY$ of $\mathbb{H}_g$ with $X = \mathrm{Re}\,\Omega$ and $Y = \mathrm{Im}\,\Omega$. For a pair (A, B) with $A, B \in \mathbb{Z}^g$, we define the function $E_{\Omega;A,B} : \mathbb{C}^g \longrightarrow \mathbb{C}$ by

$$E_{\Omega;A,B}(Z) = e^{2\pi i\left(\sigma({}^tAU) + \sigma((B-AX)Y^{-1}\,{}^tV)\right)},$$

where $Z = U + iV$ is a variable in $\mathbb{C}^g$ with real U, V.

Lemma 4.1. *For any $A, B \in \mathbb{Z}^g$, the function $E_{\Omega;A,B}$ satisfies the following functional equation*

$$E_{\Omega;A,B}(Z + \lambda\Omega + \mu) = E_{\Omega;A,B}(Z), \quad Z \in \mathbb{C}^g$$

for all $\lambda, \mu \in \mathbb{Z}^g$. Thus $E_{\Omega;A,B}$ can be regarded as a function on A_Ω.

Proof. The proof can be found in [157]. □

We let $L^2(A_\Omega)$ be the space of all functions $f : A_\Omega \longrightarrow \mathbb{C}$ such that

$$||f||_\Omega := \int_{A_\Omega} |f(Z)|^2 dv_\Omega,$$

where dv_Ω is the volume element on A_Ω normalized so that $\int_{A_\Omega} dv_\Omega = 1$. The inner product $(\ ,\ )_\Omega$ on the Hilbert space $L^2(A_\Omega)$ is given by

$$(f, g)_\Omega := \int_{A_\Omega} f(Z)\,\overline{g(Z)}\,dv_\Omega, \quad f, g \in L^2(A_\Omega).$$

Theorem 4.1. *The set $\{\, E_{\Omega;A,B} \mid A, B \in \mathbb{Z}^g \,\}$ is a complete orthonormal basis for $L^2(A_\Omega)$. Moreover we have the following spectral decomposition of Δ_Ω:*

$$L^2(A_\Omega) = \bigoplus_{A,B \in \mathbb{Z}^g} \mathbb{C} \cdot E_{\Omega;A,B}.$$

Proof. The complete proof can be found in [157]. □

5. **Siegel Modular Forms**

5.1. Basic Properties of Siegel Modular Forms

Let ρ be a rational representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ.

Definition. A holomorphic function $f : \mathbb{H}_g \longrightarrow V_\rho$ is called a *Siegel modular form* with respect to ρ if

$$f(\gamma\cdot\Omega) = f\big((A\Omega+B)(C\Omega+D)^{-1}\big) = \rho(C\Omega+D)f(\Omega) \tag{5.1}$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and all $\Omega \in \mathbb{H}_g$. Moreover if $g = 1$, we require that f is holomorphic at the cusp ∞.

We denote by $M_\rho(\Gamma_g)$ the vector space of all Siegel modular forms with respect to Γ_g. If $\rho = \det^k$ for $k \in \mathbb{Z}$, a Siegel modular form f with respect to ρ satisfies the condition

$$f(\gamma\cdot\Omega) = \det(C\Omega+D)^k\, f(\Omega), \tag{5.2}$$

where γ and Ω are as above. In this case f is called a (classical) Siegel modular form on $\mathbb{H}_g$ of weight k. We denote by $M_k(\Gamma_g)$ the space of all Siegel modular forms on $\mathbb{H}_g$ of weight k.

Remark. (1) If $\rho = \rho_1 \oplus \rho_2$ is a direct sum of two finite dimensional rational representations of $GL(g,\mathbb{C})$, then it is easy to see that $M_\rho(\Gamma_g)$ is isomorphic to $M_{\rho_1}(\Gamma_g) \oplus M_{\rho_1}(\Gamma_g)$. Therefore it suffices to study $M_\rho(\Gamma_g)$ for an irreducible representation ρ of $GL(g,\mathbb{C})$.
(2) We may equip V_ρ with a hermitian inner product $(\ ,\ )$ satisfying the following condition

$$\big(\rho(x)v_1, v_2\big) = \big(v_1, \overline{\rho({}^t x)}v_2\big), \quad x \in GL(g,\mathbb{C}),\ v_1, v_2 \in V_\rho. \tag{5.3}$$

For an irreducible finite dimensional representation (ρ, V_ρ) of $GL(g,\mathbb{C})$, there exist a highest weight $k(\rho) = (k_1, \cdots, k_g) \in \mathbb{Z}^g$ with $k_1 \geq \cdots \geq k_g$ and a highest weight vector $v_\rho (\neq 0) \in V_\rho$ such that

$$\rho\big(\mathrm{diag}(a_1, \cdots, a_g)\big)v_\rho = \prod_{i=1}^g a_i^{k_i}\, v_\rho, \quad a_1, \cdots, a_g \in \mathbb{C}^\times.$$

Such a vector v_ρ is uniquely determined up to scalars. The number $k(\rho) := k_g$ is called the *weight* of ρ. For example, if $\rho = \det^k$, its highest weight is $(k, k, \cdots, k)$ and hence its weight is k.

Assume that (ρ, V_ρ) is an irreducible finite dimensional rational representation of $GL(g,\mathbb{C})$. Then it is known [65, 92] that a Siegel modular form f in $M_\rho(\Gamma_g)$ admits a Fourier expansion

$$f(\Omega) = \sum_{T\geq 0} a(T)\, e^{2\pi i\, \sigma(T\Omega)}, \tag{5.4}$$

where T runs over the set of all half-integral semi-positive symmetric matrices of degree g. We recall that T is said to be *half-integral* if $2T$ is an integral matrix whose diagonal entries are even.

Theorem 5.1. *(1) If kg is odd, then $M_k(\Gamma_g) = 0$.*
(2) If $k < 0$, then $M_k(\Gamma_g) = 0$.
(3) Let ρ be a non-trivial irreducible finite dimensional representation of $GL(g,\mathbb{C})$ with

highest weight $(k_1, \cdots, k_g)$. *If* $M_\rho(\Gamma_g) \neq \{0\}$, *then* $k_g \geq 1$.
(4) If $f \in M_\rho(\Gamma_g)$, *then* f *is bounded in any subset* $\mathcal{H}(c)$ *of* $\mathbb{H}_g$ *given by the form*

$$\mathcal{H}(c) := \{\Omega \in \mathbb{H}_g \mid Im\,\Omega > c\, I_g\}$$

with any positive real number $c > 0$.

5.2. The Siegel Operator

Let (ρ, V_ρ) be an irreducible finite dimensional representation of $GL(g, \mathbb{C})$. For any positive integer r with $0 \leq r < g$, we define the operator $\Phi_{\rho,r}$ on $M_\rho(\Gamma_g)$ by

$$(5.5) \qquad \big(\Phi_{\rho,r} f\big)(\Omega_1) := \lim_{t \longrightarrow \infty} f\left(\begin{pmatrix} \Omega_1 & 0 \\ 0 & itI_{g-r} \end{pmatrix}\right), \quad f \in M_\rho(\Gamma_g),\ \Omega_1 \in \mathbb{H}_r.$$

We see that $\Phi_{\rho,r}$ is well-defined because the limit of the right hand side of (5.5) exists (cf. Theorem 5.1. (4)). The operator $\Phi_{\rho,r}$ is called the *Siegel operator*. A Siegel modular form $f \in M_\rho(\Gamma_g)$ is said to be a *cusp form* if $\Phi_{\rho,g-1} f = 0$. We denote by $S_\rho(\Gamma_g)$ the vector space of all cusp forms on $\mathbb{H}_g$ with respect to ρ. Let $V_\rho^{(r)}$ be the subspace of V_ρ spanned by the values

$$\left\{ \big(\Phi_{\rho,r} f\big)(\Omega_1) \mid \Omega_1 \in \mathbb{H}_r,\ \ f \in M_\rho(\Gamma_g) \right\}.$$

According to [143], $V_\rho^{(r)}$ is invariant under the action of the subgroup

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & I_{g-r} \end{pmatrix} \,\middle|\, a \in GL(r, \mathbb{C}) \right\}.$$

Then we have an irreducible rational representation $\rho^{(r)}$ of $GL(r, \mathbb{C})$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & I_{g-r} \end{pmatrix}\right) v, \quad a \in GL(r, \mathbb{C}),\ v \in V_\rho^{(r)}.$$

We observe that if $(k_1, \cdots, k_g)$ is the highest weight of ρ, then $(k_1, \cdots, k_r)$ is the highest weight of $\rho^{(r)}$.

Theorem 5.2. *The Siegel operator* $\Phi_{\det^k, r} : M_k(\Gamma_g) \longrightarrow M_k(\Gamma_r)$ *is surjective for* k *even with* $k > \frac{g+r+3}{2}$.

The proof of Theorem 5.2 can be found in [144].

We define the Petersson inner product $\langle\ ,\ \rangle_P$ on $M_\rho(\Gamma_g)$ by

$$(5.6) \qquad \langle f_1, f_2 \rangle_P := \int_{\mathcal{F}_g} \big(\rho(\operatorname{Im}\Omega) f_1(\Omega), f_2(\Omega)\big)\, dv_g(\Omega), \quad f_1, f_2 \in M_\rho(\Gamma_g),$$

where $\mathcal{F}_g$ is the Siegel's fundamental domain, $(\ ,\ )$ is the hermitian inner product defined in (5.3) and $dv_g(\Omega)$ is the volume element defined by (2.3). We can check that the integral of (5.6) converges absolutely if one of f_1 and f_2 is a cusp form. It is easily seen that one has the orthogonal decomposition

$$M_\rho(\Gamma_g) = S_\rho(\Gamma_g) \oplus S_\rho(\Gamma_g)^\perp,$$

where

$$S_\rho(\Gamma_g)^\perp = \left\{ f \in M_\rho(\Gamma_g) \,|\, \langle f, h\rangle_P = 0 \text{ for all } h \in S_\rho(\Gamma_g) \right\}$$

is the orthogonal complement of $S_\rho(\Gamma_g)$ in $M_\rho(\Gamma_g)$.

5.3. Construction of Siegel Modular Forms

In this subsection, we provide several well-known methods to construct Siegel modular forms.

(A) Klingen's Eisenstein Series

Let r be an integer with $0 \le r < g$. We assume that k is a positive *even* integer. For $\Omega \in \mathbb{H}_g$, we write

$$\Omega = \begin{pmatrix} \Omega_1 & * \\ * & \Omega_2 \end{pmatrix}, \quad \Omega_1 \in \mathbb{H}_r,\ \Omega_2 \in \mathbb{H}_{g-r}.$$

For a fixed *cusp* form $f \in S_k(\Gamma_r)$ of weight k, H. Klingen [73] introduced the Eisenstein series $E_{g,r,k}(f)$ formally defined by

(5.7)

$$E_{g,r,k}(f)(\Omega) := \sum_{\gamma \in P_r \backslash \Gamma_g} f\big((\gamma\cdot\Omega)_1\big)\cdot \det(C\Omega + D)^{-k}, \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g,$$

where

$$P_r = \left\{ \begin{pmatrix} A_1 & 0 & B_1 & * \\ * & U & * & * \\ C_1 & 0 & D_1 & * \\ 0 & 0 & 0 & {}^tU^{-1} \end{pmatrix} \in \Gamma_g \;\middle|\; \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \Gamma_r,\ U \in GL(g-r, \mathbb{Z}) \right\}$$

is a parabolic subgroup of Γ_g. We note that if $r = 0$, and if $f = 1$ is a constant, then

$$E_{g,0,k}(\Omega) = \sum_{C,D} \det(C\Omega + D)^{-k},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over the set of all representatives for the cosets $GL(g,\mathbb{Z})\backslash\Gamma_g$.

Klingen [73] proved the following:

Theorem 5.3. *Let $g \ge 1$ and let r be an integer with $0 \le r < g$. We assume that k is a positive even integer with $k > g + r + 1$. Then for any cusp form $f \in S_k(\Gamma_r)$ of weight k,the Eisenstein series $E_{g,r,k}(f)$ converges to a Siegel modular form on $\mathbb{H}_g$ of the same weight k and one has the following property*

$$\Phi_{\det^k,\, r} E_{g,r,k}(f) = f. \tag{5.8}$$

The proof of the above theorem can be found in [73, 74, 92].

(B) Theta Series

Let (ρ, V_ρ) be a finite dimensional rational representation of $GL(g,\mathbb{C})$. We let $H_\rho(r,g)$ be the space of pluriharmonic polynomials $P : \mathbb{C}^{(r,g)} \longrightarrow V_\rho$ with respect to (ρ, V_ρ). That is, $P \in H_\rho(r,g)$ if and only if $P : \mathbb{C}^{(r,g)} \longrightarrow V_\rho$ is a V_ρ-valued polynomial on $\mathbb{C}^{(r,g)}$ satisfying the following conditions (5.9) and (5.10): if $z=(z_{kj})$ is a coordinate in $\mathbb{C}^{(r,g)}$,

$$\sum_{k=1}^{r}\frac{\partial^2 P}{\partial z_{ki}\partial z_{kj}}=0 \quad \text{for all } i,j \text{ with } 1\le i,j\le g \tag{5.9}$$

and

$$P(zh)=\rho({}^t h)\,\det(h)^{-\frac{r}{2}}P(z) \quad \text{for all } z\in\mathbb{C}^{(r,g)} \text{ and } h\in GL(g,\mathbb{C}). \tag{5.10}$$

Now we let S be a positive definite even unimodular matrix of degree r. To a pair (S,P) with $P\in H_\rho(r,g)$, we attach the theta series

$$\Theta_{S,P}(\Omega):=\sum_{A\in\mathbb{Z}^{(r,g)}}P(S^{\frac{1}{2}}A)\,e^{\pi i\,\sigma(S[A]\Omega)} \tag{5.11}$$

which converges for all $\Omega\in\mathbb{H}_g$. E. Freitag [34] proved that $\Theta_{S,P}$ is a Siegel modular form on $\mathbb{H}_g$ with respect to ρ, i.e., $\Theta_{S,P}\in M_\rho(\Gamma_g)$.

Next we describe a method of constructing Siegel modular forms using the so-called *theta constants.*

We consider a theta characteristic

$$\epsilon=\begin{pmatrix}\epsilon'\\ \epsilon''\end{pmatrix}\in\{0,1\}^{2g} \quad \text{with} \quad \epsilon',\epsilon''\in\{0,1\}^g.$$

A theta characteristic $\epsilon=\begin{pmatrix}\epsilon'\\ \epsilon''\end{pmatrix}$ is said to be *odd* (resp. *even*) if ${}^t\epsilon'\epsilon''$ is odd (resp. even). Now to each theta characteristic $\epsilon=\begin{pmatrix}\epsilon'\\ \epsilon''\end{pmatrix}$, we attach the theta series

$$\theta[\epsilon](\Omega):=\sum_{m\in\mathbb{Z}^g}e^{\pi i\left\{\Omega\left[m+\frac{1}{2}\epsilon'\right]+{}^t\left(m+\frac{1}{2}\epsilon'\right)\epsilon''\right\}},\quad \Omega\in\mathbb{H}_g. \tag{5.12}$$

If ϵ is odd, we see that $\theta[\epsilon]$ vanishes identically. If ϵ is even, $\theta[\epsilon]$ is a Siegel modular form on $\mathbb{H}_g$ of weight $\frac{1}{2}$ with respect to the principal congreuence subgroup $\Gamma_g(2)$ (cf. [65, 103]). Here

$$\Gamma_g(2)=\left\{\,\sigma\in\Gamma_g \mid \sigma\equiv I_{2g}\ (\mathrm{mod}\ 2)\,\right\}$$

is a congruence subgroup of Γ_g of level 2. These theta series $\theta[\epsilon]$ are called *theta constants.* It is easily checked that there are $2^{g-1}(2^g+1)$ even theta characteristics. These theta constants $\theta[\epsilon]$ can be used to construct Siegel modular forms with respect to Γ_g. We provide several examples. For $g=1$, we have

$$\left(\theta[\epsilon_{00}]\,\theta[\epsilon_{01}]\,\theta[\epsilon_{11}]\right)^8\in S_{12}(\Gamma_1),$$

where

$$\epsilon_{00}=\begin{pmatrix}0\\0\end{pmatrix},\quad \epsilon_{01}=\begin{pmatrix}0\\1\end{pmatrix}\quad\text{and}\quad \epsilon_{11}=\begin{pmatrix}1\\1\end{pmatrix}.$$

For $g=2$, we get

$$\chi_{10}:=-2^{-14}\prod_{\epsilon\in\mathbb{E}}\theta[\epsilon]^2\in S_{10}(\Gamma_2)$$

and

$$\left(\prod_{\epsilon\in\mathbb{E}}\theta[\epsilon]\right)\cdot\sum_{\epsilon_1,\epsilon_2,\epsilon_3}\left(\theta[\epsilon_1]\,\theta[\epsilon_2]\,\theta[\epsilon_3]\right)^{20}\in S_{35}(\Gamma_2),$$

where $\mathbb{E}$ denotes the set of all even theta characteristics and $(\epsilon_1,\epsilon_2,\epsilon_3)$ runs over the set of triples of theta characteristics such that $\epsilon_1+\epsilon_2+\epsilon_3$ is odd. For $g=3$, we have

$$\prod_{\epsilon\in\mathbb{E}}\theta[\epsilon]\in S_{18}(\Gamma_3).$$

We refer to [65] for more details.

5.4. Singular Modular Forms

We know that a Siegel modular form $f\in M_\rho(\Gamma_g)$ has a Fourier expansion

$$f(\Omega)=\sum_{T\geq 0}a(T)\,e^{2\pi i\,\sigma(T\Omega)},$$

where T runs over the set of all half-integral semi-positive symmetric matrices of degree g. A Siegel modular form $f\in M_\rho(\Gamma_g)$ is said to be *singular* if $a(T)\neq 0$ implies $\det(T)=0$. We observe that the notion of singular modular forms is opposite to that of cusp forms. Obviously if $g=1$, singular modular forms are constants.

We now characterize singular modular forms in terms of the weight of ρ and a certain differential operator. For a coordinate $\Omega=X+iY$ in $\mathbb{H}_g$ with X real and $Y=(y_{ij})\in\mathcal{P}_g$ (cf. Section 4), we define the differential

$$M_g:=\det(Y)\cdot\det\left(\frac{\partial}{\partial Y}\right)\tag{5.13}$$

which is invariant under the action (4.1) of $GL(g,\mathbb{R})$. Here

$$\frac{\partial}{\partial Y}=\left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial y_{ij}}\right).$$

Using the differential operator M_g, Maass [92, pp. 202-204] proved that if a nonzero singular modular form on $\mathbb{H}_g$ of weight k exists, then $nk\equiv 0\,(\mathrm{mod}\ 2)$ and $0<2k\leq g-1$. The converse was proved by Weissauer (cf. [143, Satz 4]).

Theorem 5.4. *Let ρ be an irreducible rational finite dimensional representation of $GL(g,\mathbb{C})$ with highest weight $(k_1,\cdots,k_g)$. Then a non-zero Siegel modular form $f\in M_\rho(\Gamma_g)$ is singular if and only if $2k(\rho)=2k_g<g$.*

The above theorem was proved by Freitag [33], Weissauer [143] et al. By Theorem 5.6, we see that the weight of a singular modular form is small. For instance, W. Duke and Ö. Imamoḡlu [27] proved that $S_6(\Gamma_g)=0$ for all g. In a sense we say that there are *no* cusp forms of *small weight*.

Theorem 5.5. *Let $f\in M_\rho(\Gamma_g)$ be a Siegel modular form with respect to a rational representation ρ of $GL(g,\mathbb{C})$. Then the following are equivalent:*

(1) f is a singular modular form.

(2) f satisfies the differential equation $M_g f=0$.

We refer to [92] and [152] for the proof.

Let $f \in M_k(\Gamma_g)$ be a nonzero singular modular form of weight k. According to Theorem 5.4, $2k < g$. We can show that k is divisible by 4. Let $S_1, \cdots, S_h$ be a complete system of representatives of positive definite even unimodular integral matrices of degree $2k$. Freitag [33, 34] proved that $f(\Omega)$ can be written as a linear combination of theta series $\theta_{S_1}, \cdots, \theta_{S_h}$, where θ_{S_ν} $(1 \leq \nu \leq h)$ is defined by

$$\theta_{S_\nu}(\Omega) := \sum_{A \in \mathbb{Z}^{(2k,g)}} e^{\pi i\, \sigma(S_\nu[A]\Omega)}, \quad 1 \leq \nu \leq h. \tag{5.14}$$

According to Theorem 5.5, we need to investigate some properties of the weight of ρ in order to understand singular modular forms. Let $(k_1, \cdots, k_g)$ be the highest weight of ρ. We define the *corank* of ρ by

$$\mathrm{corank}(\rho) := \left| \left\{ j \mid 1 \leq j \leq g,\ k_j = k_g \right\} \right|.$$

Let

$$f(\Omega) = \sum_{T \geq 0} a(T)\, e^{2\pi i\, \sigma(T\Omega)}$$

be a Siegel modular form in $M_\rho(\Gamma_g)$. The notion of the *rank* of f and that of the *corank* of f were introduced by Weissauer [143] as follows:

$$\mathrm{rank}(f) := \max\left\{ \mathrm{rank}\,(T) \mid a(T) \neq 0 \right\}$$

and

$$\mathrm{corank}(f) := g - \min\left\{ \mathrm{rank}\,(T) \mid a(T) \neq 0 \right\}.$$

Weissauer [143] proved the following.

Theorem 5.6. *Let ρ be an irreducible rational representation of $GL(g, \mathbb{C})$ with highest weight $(k_1, \cdots, k_g)$ such that corank$(\rho) < g - k_g$. Assume that*

$$\left| \left\{ j \mid 1 \leq j \leq g,\ k_j = k_g + 1 \right\} \right| < 2\big(g - k_g - \mathit{corank}(\rho)\big).$$

Then $M_\rho(\Gamma_g) = 0$.

6. The Hecke Algebra

6.1. The Structure of the Hecke Algebra

For a positive integer g, we let $\Gamma_g = Sp(g, \mathbb{Z})$ and let

$$\Delta_g := GSp(g, \mathbb{Q}) = \left\{ M \in GL(2g, \mathbb{Q}) \mid {}^tMJ_gM = l(M)J_g,\ l(M) \in \mathbb{Q}^\times \right\}$$

be the group of symplectic similitudes of the rational symplectic vector space $(\mathbb{Q}^{2g}, \langle\ ,\ \rangle)$. We put

$$\Delta_g^+ := GSp(g, \mathbb{Q})^+ = \left\{ M \in \Delta_g \mid l(M) > 0 \right\}.$$

Following the notations in [34], we let $\mathscr{H}(\Gamma_g, \Delta_g)$ be the complex vector space of all formal finite sums of double cosets $\Gamma_g M \Gamma_g$ with $M \in \Delta_g^+$. A double coset $\Gamma_g M \Gamma_g\,(M \in \Delta_g^+)$ can be written as a finite disjoint union of right cosets $\Gamma_g M_\nu\,(1 \leq \nu \leq h)$:

$$\Gamma_g M \Gamma_g = \cup_{\nu=1}^h \Gamma_g M_\nu \quad \text{(disjoint)}.$$

Let $\mathscr{L}(\Gamma_g, \Delta_g)$ be the complex vector space consisting of formal finite sums of right cosets $\Gamma_g M$ with $M \in \Delta^+$. For each double coset $\Gamma_g M \Gamma_g = \cup_{\nu=1}^h \Gamma_g M_\nu$ we associate an element $j(\Gamma_g M \Gamma_g)$ in $\mathscr{L}(\Gamma_g, \Delta_g)$ defined by

$$j(\Gamma_g M \Gamma_g) := \sum_{\nu=1}^h \Gamma_g M_\nu.$$

Then j induces a linear map

$$j_* : \mathscr{H}(\Gamma_g, \Delta_g) \longrightarrow \mathscr{L}(\Gamma_g, \Delta_g). \tag{6.1}$$

We observe that Δ_g acts on $\mathscr{L}(\Gamma_g, \Delta_g)$ as follows:

$$\big(\sum_{j=1}^h c_j\,\Gamma_g M_j\big) \cdot M = \sum_{j=1}^h c_j\,\Gamma_g M_j M, \quad M \in \Delta_g.$$

We denote

$$\mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g} := \big\{\, T \in \mathscr{L}(\Gamma_g, \Delta_g) \mid T \cdot \gamma = T \text{ for all } \gamma \in \Gamma_g \,\big\}$$

be the subspace of Γ_g-invariants in $\mathscr{L}(\Gamma_g, \Delta_g)$. Then we can show that $\mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g}$ coincides with the image of j_* and the map

$$j_* : \mathscr{H}(\Gamma_g, \Delta_g) \longrightarrow \mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g} \tag{6.2}$$

is an isomorphism of complex vector spaces (cf. [34, p. 228]). From now on we identify $\mathscr{H}(\Gamma_g, \Delta_g)$ with $\mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g}$.

We define the multiplication of the double coset $\Gamma_g M \Gamma_g$ and $\Gamma_g N$ by

$$(\Gamma_g M \Gamma_g) \cdot (\Gamma_g N) = \sum_{j=1}^h \Gamma_g M_j N, \quad M, N \in \Delta_g, \tag{6.3}$$

where $\Gamma_g M \Gamma_g = \cup_{j=1}^h \Gamma_g M_j$ (disjoint). The definition (6.3) is well defined, i.e., independent of the choice of M_j and N. We extend this multiplication to $\mathscr{H}(\Gamma_g, \Delta_g)$ and $\mathscr{L}(\Gamma_g, \Delta_g)$. Since

$$\mathscr{H}(\Gamma_g, \Delta_g) \cdot \mathscr{H}(\Gamma_g, \Delta_g) \subset \mathscr{H}(\Gamma_g, \Delta_g),$$

$\mathscr{H}(\Gamma_g, \Delta_g)$ is an associative algebra with the identity element $\Gamma_g I_{2g} \Gamma_g = \Gamma_g$. The algebra $\mathscr{H}(\Gamma_g, \Delta_g)$ is called the *Hecke algebra* with respect to Γ_g and Δ_g.

We now describe the structure of the Hecke algebra $\mathscr{H}(\Gamma_g, \Delta_g)$. For a prime p, we let $\mathbb{Z}[1/p]$ be the ring of all rational numbers of the form $a \cdot p^\nu$ with $a, \nu \in \mathbb{Z}$. For a prime p, we denote

$$\Delta_{g,p} := \Delta_g \cap GL\big(2g, \mathbb{Z}[1/p]\big).$$

Then we have a decomposition of $\mathscr{H}(\Gamma_g, \Delta_g)$

$$\mathscr{H}(\Gamma_g, \Delta_g) = \bigotimes_{p\,:\,\text{prime}} \mathscr{H}(\Gamma_g, \Delta_{g,p})$$

as a tensor product of local Hecke algebras $\mathscr{H}(\Gamma_g, \Delta_{g,p})$. We denote by $\check{\mathscr{H}}(\Gamma_g, \Delta_g)$ (resp. $\check{\mathscr{H}}(\Gamma_g, \Delta_{g,p})$ the subring of $\mathscr{H}(\Gamma_g, \Delta_g)$ (resp. $\mathscr{H}(\Gamma_g, \Delta_{g,p})$ by integral matrices.

In order to describe the structure of local Hecke operators $\mathscr{H}(\Gamma_g, \Delta_{g,p})$, we need the following lemmas.

Lemma 6.1. *Let $M \in \Delta_g^+$ with ${}^tMJ_gM = lJ_g$. Then the double coset $\Gamma_g M \Gamma_g$ has a unique representative of the form*

$$M_0 = diag(a_1, \cdots, a_g, d_1, \cdots, d_g),$$

where $a_g | d_g,\ a_j > 0,\ a_j d_j = l$ for $1 \le j \le g$ and $a_k | a_{k+1}$ for $1 \le k \le g-1$.

For a positive integer l, we let

$$O_g(l) := \left\{ M \in GL(2g, \mathbb{Z}) \mid {}^tMJ_gM = lJ_g \right\}.$$

Then we see that $O_g(l)$ can be written as a finite disjoint union of double cosets and hence as a finite union of right cosets. We define $T(l)$ as the element of $\mathscr{H}(\Gamma_g, \Delta_g)$ defined by $O_g(l)$.

Lemma 6.2. *(a) Let l be a positive integer. Let*

$$O_g(l) = \cup_{\nu=1}^h \Gamma_g M_\nu \quad (disjoint)$$

be a disjoint union of right cosets $\Gamma_g M_\nu\,(1 \le \nu \le h)$. Then each right coset $\Gamma_g M_\nu$ has a representative of the form

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu \\ 0 & D_\nu \end{pmatrix}, \quad {}^tA_\nu D_\nu = lI_g, \quad A_\nu \text{ is upper triangular.}$$

(b) Let p be a prime. Then

$$T(p) = O_g(p) = \Gamma_g \begin{pmatrix} I_g & 0 \\ 0 & pI_g \end{pmatrix} \Gamma_g$$

and

$$T(p^2) = \sum_{i=0}^{g} T_i(p^2),$$

where

$$T_k(p^2) := \begin{pmatrix} I_{g-k} & 0 & 0 & 0 \\ 0 & pI_k & 0 & 0 \\ 0 & 0 & p^2 I_{g-k} & 0 \\ 0 & 0 & 0 & pI_k \end{pmatrix} \Gamma_g, \quad 0 \le k \le g.$$

Proof. The proof can be found in [34, p. 225 and p. 250]. □

For example, $T_g(p^2) = \Gamma_g(pI_{2g})\Gamma_g$ and

$$T_0(p^2) = \Gamma_g \begin{pmatrix} I_g & 0 \\ 0 & p^2 I_g \end{pmatrix} \Gamma_g = T(p)^2.$$

We have the following

Theorem 6.1. *The local Hecke algebra $\check{\mathscr{H}}(\Gamma_g, \Delta_{g,p})$ is generated by algebraically independent generators $T(p), T_1(p^2), \cdots, T_g(p^2)$.*

Proof. The proof can be found in [34, p. 250 and p. 261]. □

On Δ_g we have the anti-automorphism $M \mapsto M^* := l(M)M^{-1}\,(M \in \Delta_g)$. Obviously $\Gamma_g^* = \Gamma_g$. By Lemma 6.1, $(\Gamma_g M \Gamma_g)^* = \Gamma_g M^* \Gamma_g = \Gamma_g M \Gamma_g$. According to [125], Proposition 3.8, $\mathscr{H}(\Gamma_g, \Delta_g)$ is commutative.

Let $X_0, X_1, \cdots, X_g$ be the $g+1$ variables. We define the automorphisms

$$w_j : \mathbb{C}\big[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_g^{\pm 1}\big] \longrightarrow \mathbb{C}\big[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_g^{\pm 1}\big], \quad 1 \leq j \leq g$$

by

$$w_j(X_0) = X_0 X_j^{-1}, \quad w_j(X_j) = X_j^{-1}, \quad w_j(X_k) = X_k \ \text{ for } k \neq 0, j.$$

Let W_g be the finite group generated by $w_1, \cdots, w_g$ and the permutations of variables $X_1, \cdots, X_g$. Obviously w_j^2 is the identity map and $|W_g| = 2^g g!$.

Theorem 6.2. *There exists an isomorphism*

$$Q : \mathscr{H}(\Gamma_g, \Delta_{g,p}) \longrightarrow \mathbb{C}\big[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_g^{\pm 1}\big]^{W_g}.$$

In fact, Q is defined by

$$Q\Big(\sum_{j=1}^{h} \Gamma_g M_j\Big) = \sum_{j=1}^{h} Q(\Gamma_g M_j) = \sum_{j=1}^{h} X_0^{-k_0(j)} \prod_{\nu=1}^{g} \big(p^{-\nu} X^{\nu}\big)^{k_\nu(j)} |\det A_j|^{g+1},$$

where we choose the representative M_j of $\Gamma_g M_j$ of the form

$$M_j = \begin{pmatrix} A_j & B_j \\ 0 & D_j \end{pmatrix}, \quad A_j = \begin{pmatrix} p^{k_1(j)} & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & p^{k_g(j)} \end{pmatrix}.$$

We note that the integers $k_1(j), \cdots, k_g(j)$ are uniquely determined.

Proof. The proof can be found in [34]. □

For a prime p, we let

$$\mathscr{H}(\Gamma_g, \Delta_{g,p})_{\mathbb{Q}} := \Big\{ \sum c_j \, \Gamma_g M_j \Gamma_g \in \mathscr{H}(\Gamma_g, \Delta_{g,p}) \mid c_j \in \mathbb{Q} \Big\}$$

be the $\mathbb{Q}$-algebra contained in $\mathscr{H}(\Gamma_g, \Delta_{g,p})$. We put

$$G_p := GSp(g, \mathbb{Q}_p) \quad \text{and} \quad K_p = GSp(g, \mathbb{Z}_p).$$

We can identify $\mathscr{H}(\Gamma_g, \Delta_{g,p})_{\mathbb{Q}}$ with the $\mathbb{Q}$-algebra $\mathscr{H}_{g,p}^{\mathbb{Q}}$ of $\mathbb{Q}$-valued locally constant, K_p-biinvariant functions on G_p with compact support. The multiplication on $\mathscr{H}_{g,p}^{\mathbb{Q}}$ is given by

$$(f_1 * f_2)(h) = \int_{G_p} f_1(g)\, f_2(g^{-1}h) dg, \quad f_1, f_2 \in \mathscr{H}_{g,p}^{\mathbb{Q}},$$

where dg is the unique Haar measure on G_p such that the volume of K_p is 1. The correspondence is obtained by sending the double coset $\Gamma_g M \Gamma_g$ to the characteristic function of $K_p M K_p$.

In order to describe the structure of $\mathscr{H}_{g,p}^{\mathbb{Q}}$, we need to understand the p-adic Hecke algebras of the diagonal torus $\mathbb{T}$ and the Levi subgroup $\mathbb{M}$ of the standard parabolic group. Indeed, $\mathbb{T}$ is defined to be the subgroup consisting of diagonal matrices in Δ_g and

$$\mathbb{M} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \Delta_g \right\}$$

is the Levi subgroup of the parabolic subgroup

$$\left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Delta_g \right\}.$$

Let Y be the co-character group of $\mathbb{T}$, i.e., $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$. We define the local Hecke algebra $\mathscr{H}_p(\mathbb{T})$ for $\mathbb{T}$ to be the $\mathbb{Q}$-algebra of $\mathbb{Q}$-valued, $\mathbb{T}(\mathbb{Z}_p)$-biinvariant functions on $\mathbb{T}(\mathbb{Q}_p)$ with compact support. Then $\mathscr{H}_p(\mathbb{T}) \cong \mathbb{Q}[Y]$, where $\mathbb{Q}[Y]$ is the group algebra over $\mathbb{Q}$ of Y. An element $\lambda \in Y$ corresponds the characteristic function of the double coset $D_\lambda = K_p \lambda(p) K_p$. It is known that $\mathscr{H}_p(\mathbb{T})$ is isomorphic to the ring $\mathbb{Q}\big[(u_1/v_1)^{\pm 1}, \cdots, (u_g/v_g)^{\pm 1}, (v_1 \cdots v_g)^{\pm 1}\big]$ under the map

$$(a_1, \cdots, a_g, c) \mapsto (u_1/v_1)^{a_1} \cdots (u_g/v_g)^{a_g} (v_1 \cdots v_g)^c.$$

Similarly we have a p-adic Hecke algebra $\mathscr{H}_p(\mathbb{M})$. Let $W_{\Delta_g} = N(\mathbb{T})/\mathbb{T}$ be the Weyl group with respect to $(\mathbb{T}, \Delta_g)$, where $N(\mathbb{T})$ is the normalizer of $\mathbb{T}$ in Δ_g. Then $W_{\Delta_g} \cong S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$, where the generator of the i-th factor $\mathbb{Z}/2\mathbb{Z}$ acts on a matrix of the form $\mathrm{diag}(a_1, \cdots, a_g, d_1, \cdots, d_g)$ by interchanging a_i and d_i, and the symmetry group S_g acts by permuting the a_i*'s* and d_i*'s.* We note that W_{Δ_g} is isomorphic to W_g. The Weyl group $W_{\mathbb{M}}$ with respect to $(\mathbb{T}, \mathbb{M})$ is isomorphic to S_g. We can prove that the algebra $\mathscr{H}_p(\mathbb{T})^{W_{\Delta_g}}$ of W_{Δ_g}-invariants in $\mathscr{H}_p(\mathbb{T})$ is isomorphic to $\mathbb{Q}\big[Y_0^{\pm 1}, Y_1, \cdots, Y_g\big]$ (cf. [34]). We let

$$B = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Delta_g \,\middle|\, A \text{ is upper triangular, } D \text{ is lower triangular} \right\}$$

be the Borel subgroup of Δ_g. A set Φ^+ of positive roots in the root system Φ determined by B. We set $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Now we have the map $\alpha_{\mathbb{M}} : \mathbb{M} \longrightarrow \mathbb{G}_m$ defined by

$$\alpha_{\mathbb{M}}(M) := l(M)^{-\frac{g(g+1)}{2}} \big(\det A\big)^{g+1}, \quad M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathbb{M}$$

and the map $\beta_{\mathbb{T}} : \mathbb{T} \longrightarrow \mathbb{G}_m$ defined by

$$\beta_{\mathbb{T}}(\mathrm{diag}(a_1, \cdots, a_g, d_1, \cdots, d_g)) := \prod_{i=1}^{g} a_1^{g+1-2i}, \ \mathrm{diag}(a_1, \cdots, a_g, d_1, \cdots, d_g) \in \mathbb{T}.$$

Let $\theta_{\mathbb{T}} := \alpha_{\mathbb{M}} \beta_{\mathbb{T}}$ be the character of $\mathbb{T}$. The *Satake's spherical map* $S_{p,\mathbb{M}} : \mathscr{H}_{g,p}^{\mathbb{Q}} \longrightarrow \mathscr{H}_p(\mathbb{M})$ is defined by

$$S_{p,\mathbb{M}}(\phi)(m) := |\alpha_{\mathbb{M}}(m)|_p \int_{U(\mathbb{Q}_p)} \phi(mu) du, \quad \phi \in \mathscr{H}_{g,p}^{\mathbb{Q}}, \ m \in \mathbb{M}, \tag{6.4}$$

where $|\ \ |_p$ is the p-adic norm and $U(\mathbb{Q}_p)$ denotes the unipotent radical of Δ_g. Also another *Satake's spherical map* $S_{\mathbb{M},\mathbb{T}} : \mathscr{H}_p(\mathbb{M}) \longrightarrow \mathscr{H}_p(\mathbb{T})$ is defined by

$$S_{\mathbb{M},\mathbb{T}}(f)(t) := |\beta_{\mathbb{T}}(t)|_p \int_{\mathbb{M} \cap \mathbb{N}} f(tn) dn, \quad t \in \mathscr{H}_p(\mathbb{T}), \ t \in \mathbb{T}, \tag{6.5}$$

where $\mathbb{N}$ is a nilpotent subgroup of Δ_g.

Theorem 6.3. *The Satake's spherical maps $S_{p,\mathbb{M}}$ and $S_{\mathbb{M},\mathbb{T}}$ define the isomorphisms of $\mathbb{Q}$-algebras*

$$\mathscr{H}_{g,p}^{\mathbb{Q}} \cong \mathscr{H}_p(\mathbb{T})^{W_{\Delta_g}} \quad \textit{and} \quad \mathscr{H}_p(\mathbb{M}) \cong \mathscr{H}_p(\mathbb{T})^{W_{\mathbb{M}}}. \tag{6.6}$$

We define the elements ϕ_k $(0 \leq k \leq g)$ in $\mathscr{H}_p(\mathbb{M})$ by

$$\phi_k := p^{-\frac{k(k+1)}{2}}\, \mathbb{M}(\mathbb{Z}_p) \begin{pmatrix} I_{g-k} & 0 & 0 \\ 0 & pI_g & 0 \\ 0 & 0 & I_k \end{pmatrix} \mathbb{M}(\mathbb{Z}_p), \quad i = 0, 1, \cdots, g.$$

Then we have the relation

$$S_{p,\mathbb{M}}(T(p)) = \sum_{k=0}^{g} \phi_k \tag{6.7}$$

and

$$S_{p,\mathbb{M}}\big(T_i(p^2)\big) = \sum_{j,k\geq 0,\, i+j\leq k} m_{k-j}(i)\, p^{-\binom{k-j+1}{2}} \phi_j \phi_k, \tag{6.8}$$

where

$$m_s(i) := \sharp \left\{ A \in M(s, \mathbb{F}_p) \mid {}^tA = A, \quad \mathrm{corank}(A) = i \right\}.$$

Moreover, for $k = 0, 1, \cdots, g$, we have

$$S_{\mathbb{M},\mathbb{T}}(\phi_k) = (v_1 \cdots v_g) E_k(u_1/v_1, \cdots, u_g/v_g), \tag{6.9}$$

where E_k denotes the elementary symmetric function of degree k. The proof of (6.7)-(6.9) can be found in [2, pp. 142-145].

6.2. Action of the Hecke Algebra on Siegel Modular Forms

Let (ρ, V_ρ) be a finite dimensional irreducible representation of $GL(g, \mathbb{C})$ with highest weight $(k_1, \cdots, k_g)$. For a function $F : \mathbb{H}_g \longrightarrow V_\rho$ and $M \in \Delta_g^+$, we define

$$(f|_\rho M)(\Omega) = \rho(C\Omega + D)^{-1} f(M \cdot \Omega), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta_g^+.$$

It is easily checked that $f|_\rho M_1 M_2 = \big(f|_\rho M_1\big)|_\rho M_2$ for $M_1, M_2 \in \Delta_g^+$.

We now consider a subset $\mathscr{M}$ of Δ_g satisfying the following properties (M1) and (M2):

(M1) $\mathscr{M} = \cup_{j=1}^h \Gamma_g M_j$ (disjoint union);
(M2) $\mathscr{M}\, \Gamma_g \subset \mathscr{M}$.

For a Siegel modular form $f \in M_\rho(\Gamma_g)$, we define

$$T(\mathscr{M}) f := \sum_{j=1}^{h} f|_\rho M_j. \tag{6.10}$$

This is well defined, i.e., is independent of the choice of representatives M_j because of the condition (M1). On the other hand, it follows from the condition (M2) that $T(\mathscr{M}) f|_\rho \gamma = T(\mathscr{M}) f$ for all $\gamma \in \Gamma_g$. Thus we get a linear operator

$$T(\mathscr{M}) : M_\rho(\Gamma_g) \longrightarrow M_\rho(\Gamma_g). \tag{6.11}$$

We know that each double coset $\Gamma_g M \Gamma_g$ with $M \in \Delta_g$ satisfies the condition $(M1)$ and $(M2)$. Thus a linear operator $T(\mathscr{M})$ defined in (6.10 induces naturally the action of the Hecke algebra $\mathscr{H}(\Gamma_g, \Delta_g)$ on $M_\rho(\Gamma_g)$. More precisely, if $\mathscr{N} = \sum_{j=1}^h c_j \Gamma_g M_j \Gamma_g \in \mathscr{H}(\Gamma_g, \Delta_g)$, we define

$$T(\mathscr{N}) = \sum_{j=1}^h c_j T(\Gamma_g M_j \Gamma_g).$$

Then $T(\mathscr{N})$ is an endomorphism of $M_\rho(\Gamma_g)$.

Now we fix a Siegel modular form F in $M_\rho(\Gamma_g)$ which is an eigenform of the Hecke algebra $\mathscr{H}(\Gamma_g, \Delta_g)$. Then we obtain an algebra homomorphism $\lambda_F : \mathscr{H}(\Gamma_g, \Delta_g) \longrightarrow \mathbb{C}$ determined by

$$T(F) = \lambda_F(T)F, \quad T \in \mathscr{H}(\Gamma_g, \Delta_g).$$

By Theorem 6.2 or Theorem 6.3, one has

$$\begin{aligned} \mathscr{H}(\Gamma_g, \Delta_{g,p}) &\cong \mathscr{H}_{g,p}^{\mathbb{Q}} \otimes \mathbb{C} \cong \mathbb{C}[Y]^{W_g} \\ &\cong \mathscr{H}_p(\mathbb{T})^{W_g} \otimes \mathbb{C} \\ &\cong \mathbb{C}\big[(u_1/v_1)^{\pm 1}, \cdots, (u_g/v_g)^{\pm 1}, (v_1 \cdots v_g)^{\pm 1}\big]^{W_g} \\ &\cong \mathbb{C}[Y_0, Y_0^{-1}, Y_1, \cdots, Y_g], \end{aligned}$$

where $Y_0, Y_1, \cdots, Y_g$ are algebraically independent. Therefore one obtains an isomorphism

$$\mathrm{Hom}_{\mathbb{C}}\big(\mathscr{H}(\Gamma_g, \Delta_{g,p}), \mathbb{C}\big) \cong \mathrm{Hom}_{\mathbb{C}}\big(\mathscr{H}_{g,p}^{\mathbb{Q}} \otimes \mathbb{C}, \mathbb{C}\big) \cong (\mathbb{C}^\times)^{(g+1)}/W_g.$$

The algebra homomorphism $\lambda_F \in \mathrm{Hom}_{\mathbb{C}}\big(\mathscr{H}(\Gamma_g, \Delta_{g,p}), \mathbb{C}\big)$ is determined by the W_g-orbit of a certain $(g+1)$-tuple $\big(\alpha_{F,0}, \alpha_{F,1}, \cdots, \alpha_{F,g}\big)$ of nonzero complex numbers, called the *p-Satake parameters* of F. For brevity, we put $\alpha_i = \alpha_{F,i}$, $i = 0, 1, \cdots, g$. Therefore α_i is the image of u_i/v_i and α_0 is the image of $v_1 \cdots v_g$ under the map Θ. Each generator $w_i \in W_{\Delta_g} \cong W_g$ acts by

$$w_j(\alpha_0) = \alpha_0 \alpha_j^{-1} \quad w_j(\alpha_j) = \alpha_j^{-1}, \quad w_j(\alpha_k) = 0 \text{ if } k \neq 0, j.$$

These p-Satake parameters $\alpha_0, \alpha_1 \cdots, \alpha_g$ satisfy the relation

$$\alpha_0^2 \alpha_1 \cdots \alpha_g = p^{\sum_{i=1}^g k_i - g(g+1)/2}.$$

Formula (6.12) follows from the fact that $T_g(p^2) = \Gamma_g(pI_{2g})\Gamma_g$ is mapped to

$$p^{-g(g+1)/2}\, (v_1 \cdots v_g)^2 \prod_{i=1}^g (u_i/v_i).$$

We refer to [34, p. 258] for more detail. According to Formula (6.7)-(6.9), the eigenvalues $\lambda_F\big(T(p)\big)$ and $\lambda_F\big(T_i(p^2)\big)$ with $1 \leq i \leq g$ are given respectively by

$$\lambda_F\big(T(p)\big) = \alpha_0(1 + E_1 + E_2 + \cdots + E_g) \tag{6.12}$$

and

$$\lambda_F\big(T_i(p^2)\big) = \sum_{j,k \geq 0,\, j+i \leq k}^g m_{k-j}(i)\, p^{-\binom{k-j+1}{2}}\, \alpha_0^2 E_j E_k, \quad i = 1, \cdots, g, \tag{6.13}$$

where E_j denotes the elementary symmetric function of degree j in the variables $\alpha_1, \cdots, \alpha_g$. The point is that the above eigenvalues $\lambda_F\big(T(p)\big)$ and $\lambda_F\big(T_i(p^2)\big)$ $(1 \leq i \leq g)$ are described in terms of the p-Satake parameters $\alpha_0, \alpha_1 \cdots, \alpha_g$.

Examples. (1) Suppose $g(\tau) = \sum_{n \geq 1} a(n)\, e^{2\pi i n \tau}$ is a normalized eigenform in $S_k(\Gamma_1)$. Let p be a prime. Let β be a complex number determined by the relation

$$(1 - \beta X)(1 - \bar{\beta} X) = 1 - a(p) X + p^{k-1} X^2.$$

Then

$$\beta + \bar{\beta} = a(p) \quad \text{and} \quad \beta\bar{\beta} = p^{k-1}.$$

The p-Satake parameters α_0 and α_1 are given by

$$(\alpha_0, \alpha_1) = \left(\beta, \frac{\bar{\beta}}{\beta}\right) \quad or \quad \left(\bar{\beta}, \frac{\beta}{\bar{\beta}}\right).$$

It is easily checked that $\alpha_0^2 \alpha_1 = \beta\bar{\beta} = p^{k-1}$ (cf. Formula (6.12)).

(b) For a positive integer k with $k > g + 1$, we let

$$G_k(\Omega) := \sum_{M \in \Gamma_{g,0} \backslash \Gamma_g} \det(C\Omega + D)^k, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be the Siegel Eisenstein series of weight k in $M_k(\Gamma_g)$, where

$$\Gamma_{g,0} := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_g \right\}$$

is a parabolic subgroup of Γ_g. It is known that G_k is an eigenform of all the Hecke operators (cf. [34, p. 268]). Let $S_1, \cdots, S_h$ be a complete system of representatives of positive definite even unimodular integral matrices of degree $2k$. If $k > g + 1$, the Eisenstein series G_k can be expressed as the weighted mean of theta series $\theta_{S_1}, \cdots, \theta_{S_h}$:

$$G_k(\Omega) = \sum_{\nu=1}^{h} m_\nu\, \theta_{S_\nu}(\Omega), \quad \Omega \in \mathbb{H}_g, \tag{6.14}$$

where

$$m_\nu = \frac{A(S_\nu, S_\nu)^{-1}}{A(S_1, S_1)^{-1} + \cdots + A(S_h, S_h)^{-1}}, \quad 1 \leq \nu \leq h.$$

We recall that the theta series θ_{S_ν} is defined in Formula (5.14) and that for two symmetric integral matrices S of degree m and T of degree n, $A(S,T)$ is defined by

$$A(S,T) := \sharp\big\{\, G \in \mathbb{Z}^{(m,n)} \mid S[G] = {}^tGSG = T \,\big\}.$$

Formula (6.14) was obtained by Witt [148] as a special case of the analytic version of Siegel's Hauptsatz.

7. **Jacobi Forms**

In this section, we establish the notations and define the concept of Jacobi forms.

Let

$$Sp(g,\mathbb{R}) = \{M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \}$$

be the symplectic group of degree g, where

$$J_g := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

For two positive integers g and h, we consider the *Heisenberg group*

$$H_{\mathbb{R}}^{(g,h)} := \big\{(\lambda,\mu,\kappa) \mid \lambda,\mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric}\big\}$$

endowed with the following multiplication law

$$(\lambda,\mu,\kappa)\circ(\lambda',\mu',\kappa) := (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

We recall that the Jacobi group $G_{g,h}^J := Sp(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ is the semidirect product of the symplectic group $Sp(g,\mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ endowed with the following multiplication law

$$(M,(\lambda,\mu,\kappa))\cdot(M',(\lambda',\mu',\kappa')) := \ (MM',(\tilde\lambda+\lambda',\tilde\mu+\mu',\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda'))$$

with $M,M' \in Sp(g,\mathbb{R}), (\lambda,\mu,\kappa),\ (\lambda',\mu',\kappa') \in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde\lambda,\tilde\mu) := (\lambda,\mu)M'$. It is easy to see that $G_{g,h}^J$ acts on the Siegel-Jacobi space $\mathbb{H}_{g,h} := \mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(M,(\lambda,\mu,\kappa))\cdot(\Omega,Z) := (M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}), \tag{7.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega,Z) \in \mathbb{H}_{g,h}$.

Let ρ be a rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semi-positive definite matrix of degree h. Let $C^\infty(\mathbb{H}_{g,h},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{g,h}$ with values in V_ρ. For $f \in C^\infty(\mathbb{H}_{g,h},V_\rho)$, we define

$$\begin{aligned}
&\big(f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu,\kappa))]\big)(\Omega,Z)\\
:=\ & e^{-2\pi i\sigma(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda\Omega{}^t\lambda+2\lambda{}^tZ+(\kappa+\mu{}^t\lambda)))}\\
&\times\rho(C\Omega+D)^{-1}f(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),
\end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega,Z) \in \mathbb{H}_{g,h}$.

Definition 7.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(g,h)} := \{\,(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda,\mu \in \mathbb{Z}^{(h,g)},\ \kappa \in \mathbb{Z}^{(h,h)}\,\}.$$

Let Γ be a discrete subgroup of Γ_g of finite index. A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ is a holomorphic function $f \in C^\infty(\mathbb{H}_{g,h},V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde\gamma] = f$ for all $\tilde\gamma \in \Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$.

(B) f has a Fourier expansion of the following form :

$$f(\Omega, Z) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(g,h)}} c(T,R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma}\, \sigma(T\Omega)} \cdot e^{2\pi i \sigma(RZ)}$$

with some nonzero integer $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2}{}^t R & \mathcal{M} \end{pmatrix} \geq 0.$

If $g \geq 2$, the condition (B) is superfluous by the Köcher principle (cf. [165] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler(cf. [165] Theorem 1.8 or [29] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. For more results on Jacobi forms with $g > 1$ and $h > 1$, we refer to [112], [149]-[153] and [165].

Definition 7.2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2}{}^t R & \mathcal{M} \end{pmatrix} > 0$ for any $T,\ R$ with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2}{}^t R & \mathcal{M} \end{pmatrix} = 0.$

Example 7.3. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite, unimodular even integral matrix and $c \in \mathbb{Z}^{(2k,h)}$. We define the theta series

$$(7.2) \quad \vartheta^{(g)}_{S,c}(\Omega, Z) := \sum_{\lambda \in \mathbb{Z}^{(2k,g)}} e^{\pi i \{\sigma(S\lambda\Omega\, {}^t\lambda) + 2\sigma({}^t cS\lambda\, {}^t Z)\}}, \quad \Omega \in \mathbb{H}_g,\ Z \in \mathbb{C}^{(h,g)}.$$

We put $\mathcal{M} := \frac{1}{2}{}^t cSc$. We assume that $2k < g + rank\,(\mathcal{M})$. Then it is easy to see that $\vartheta^{(g)}_{S,c}$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$(cf. [165] p.212).

Remark. Singular Jacobi forms are characterized by a special differential operator or the weight of the associated rational representation of the general linear group $GL(g, \mathbb{C})$ (cf. [152]).

Now we will make brief historical remarks on Jacobi forms. In 1985, the names Jacobi group and Jacobi forms got kind of standard by the classic book [29] by Eichler and Zagier to remind of Jacobi's "Fundamenta nova theoriae functionum ellipticorum", which appeared in 1829 (cf. [68]). Before [29] these objects appeared more or less explicitly and under different names in the work of many authors. In 1966 Pyatetski-Shapiro [109] discussed the Fourier-Jacobi expansion of Siegel modular forms and the field of modular abelian functions. He gave the dimension of this field in the higher degree. About the same time Satake [119]-[120] introduced the notion of "groups of Harish-Chandra type" which are non reductive but still behave well enough so that he could determine their canonical automorphic factors and kernel functions. Shimura [127]-[128] gave a new foundation of the theory of complex multiplication of abelian functions using Jacobi theta functions. Kuznetsov [86] constructed functions which are almost Jacobi forms from ordinary elliptic modular functions. Starting 1981, Berndt [8]-[10] published some papers which studied the field of arithmetic Jacobi functions, ending up with a proof of Shimura reciprocity law for the field of these functions with arbitrary level. Furthermore he investigated the discrete series for the Jacobi group $G^J_{g,h}$ and developed the spectral theory for $L^2(\Gamma^J\backslash G^J_{g,h})$ in the case $g=h=1$ (cf. [11]-[13]). The connection of Jacobi forms to modular forms was given by Maass, Andrianov, Kohnen, Shimura, Eichler and Zagier. This connection is pictured as follows. For k even, we have the following isomorphisms

$$M^*_k(\Gamma_2)\,\cong\,J_{k,1}(\Gamma_1)\,\cong\,M^+_{k-\frac12}(\Gamma_0^{(1)}(4))\,\cong\,M_{2k-2}(\Gamma_1).$$

Here $M^*_k(\Gamma_2)$ denotes Maass's Spezialschar or Maass space and $M^+_{k-\frac12}(\Gamma_0^{(1)}(4))$ denotes the Kohnen plus space. These spaces shall be described in some more detail in the next section. For a precise detail, we refer to [93]-[95], [1], [29] and [76]. In 1982 Tai [134] gave asymptotic dimension formulae for certain spaces of Jacobi forms for arbitrary g and $h=1$ and used these ones to show that the moduli $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g is *of general type* for $g\geq 9$. Feingold and Frenkel [31] essentially discussed Jacobi forms in the context of Kac-Moody Lie algebras generalizing the Maass correspondence to higher level. Gritsenko [44] studied Fourier-Jacobi expansions and a non-commutative Hecke ring in connection with the Jacobi group. After 1985 the theory of Jacobi forms for $g=h=1$ had been studied more or less systematically by the Zagier school. A large part of the theory of Jacobi forms of higher degree was investigated by Kramer [82]-[83], [112], Yang [149]-[153]and Ziegler [165]. There were several attempts to establish L-functions in the context of the Jacobi group by Murase [104]-[105]and Sugano [106] using the so-called "Whittaker-Shintani functions". Kramer [82]-[83] developed an arithmetic theory of Jacobi forms of higher degree. Runge [112] discussed some part of the geometry of Jacobi forms for arbitrary g and $h=1$. For a good survey on some motivation and background for the study of Jacobi forms, we refer to [14]. The theory of Jacobi forms has been extensively studied by many people until now and has many applications in other areas like geometry and physics.

8. **Lifting of Elliptic Cusp forms to Siegel Modular Forms**

In this section, we present some results about the liftings of elliptic cusp forms to Siegel modular forms. And we discuss the Duke-Imamoğlu-Ikeda lift.

In order to discuss these lifts, we need two kinds of L-function or zeta functions associated to Siegel Hecke eigenforms. These zeta functions are defined by using the Satake parameters of their associated Siegel Hecke eigenforms.

Let $F\in M_\rho(\Gamma_g)$ be a nonzero Hecke eigenform on $\mathbb{H}_g$ of type ρ, where ρ is a finite dimensional irreducible representation of $GL(g,\mathbb{C})$ with highest weight $(k_1,\cdots,k_g)$. Let $\alpha_{p,0},\alpha_{p,1},\cdots,\alpha_{p,g}$ be the p-Satake parameters of F at a prime p. Using these Satake parameters, we define the *local spinor zeta function* $Z_{F,p}(s)$ of F at p by

$$Z_{F,p}(t)=(1-\alpha_{p,0}t)\prod_{r=1}^{g}\prod_{1\leq i_1<\cdots<i_r\leq g}(1-\alpha_{p,0}\alpha_{p,i_1}\cdots\alpha_{p,i_r}t).$$

Now we define the *spinor zeta function* $Z_F(s)$ by

$$Z_F(s) := \prod_{p\,:\,\mathrm{prime}} Z_{F,p}(p^{-s})^{-1}, \quad \mathrm{Re}\, s \gg 0. \tag{8.1}$$

For example, if $g=1$, the spinor zeta function $Z_f(s)$ of a Hecke eigenform f is nothing but the Hecke L-function $L(f,s)$ of f.

Secondly one has the so-called *standard zeta function* $D_F(s)$ of a Hecke eigenform F in $S_\rho(\Gamma_g)$ defined by

$$D_F(s) := \prod_{p\,:\,\mathrm{prime}} D_{F,p}(p^{-s})^{-1}, \tag{8.2}$$

where

$$D_{F,p}(t) = (1-t)\prod_{i=1}^{g}(1-\alpha_{p,i}t)(1-\alpha_{p,i}^{-1}t).$$

For instance, if $g=1$, the standard zeta function $D_f(s)$ of a Hecke eigenform $f(\tau)=\sum_{n=1}^\infty a(n)e^{2\pi i n\tau}$ in $S_k(\Gamma_1)$ has the following

$$D_f(s-k+1) = \prod_{p\,:\,\mathrm{prime}} \left(1+p^{-s+k-1}\right)^{-1}\cdot \sum_{n=1}^\infty a(n^2)n^{-s}.$$

For the present time being, we recall the Kohnen plus space and the Maass space. Let $\mathcal{M}$ be a positive definite, half-integral symmetric matrix of degree h. For a fixed element $\Omega\in\mathbb{H}_g$, we denote $\Theta^{(g)}_{\mathcal{M},\Omega}$ the vector space consisting of all the functions $\theta:\mathbb{C}^{(h,g)}\longrightarrow\mathbb{C}$ satisfying the condition:

$$\theta(Z+\lambda\Omega+\mu) = e^{-2\pi i\,\sigma(\mathcal{M}[\lambda]\Omega+2\,{}^tZ\mathcal{M}\lambda)}\theta(Z), \quad Z\in\mathbb{C}^{(h,g)} \tag{8.3}$$

for all $\lambda,\mu\in\mathbb{Z}^{(h,g)}$. For brevity, we put $L:=\mathbb{Z}^{(h,g)}$ and $L_{\mathcal{M}}:=L/(2\mathcal{M})L$. For each $\gamma\in L_{\mathcal{M}}$, we define the theta series $\theta_\gamma(\Omega,Z)$ by

$$\theta_\gamma(\Omega,Z) = \sum_{\lambda\in L} e^{2\pi i\,\sigma(\mathcal{M}[\lambda+(2\mathcal{M})^{-1}\gamma]\Omega+2\,{}^tZ\mathcal{M}(\lambda+(2\mathcal{M})^{-1}\gamma))},$$

where $(\Omega,Z)\in\mathbb{H}_g\times\mathbb{C}^{(h,g)}$. Then $\{\,\theta_\gamma(\Omega,Z)\mid\gamma\in L_{\mathcal{M}}\,\}$ forms a basis for $\Theta^{(g)}_{\mathcal{M},\Omega}$. For any Jacobi form $\phi(\Omega,Z)\in J_{k,\mathcal{M}}(\Gamma_g)$, the function $\phi(\Omega,\cdot)$ with fixed Ω is an element of $\Theta^{(g)}_{\mathcal{M},\Omega}$ and $\phi(\Omega,Z)$ can be written as a linear combination of theta series $\theta_\gamma(\Omega,Z)\,(\,\gamma\in L_{\mathcal{M}})$:

$$\phi(\Omega,Z) = \sum_{\gamma\in L_{\mathcal{M}}} \phi_\gamma(\Omega)\theta_\gamma(\Omega,Z). \tag{8.4}$$

We observe that $\phi=(\phi_\gamma(\Omega))_{\gamma\in L_{\mathcal{M}}}$ is a vector valued automorphic form with respect to a theta multiplier system.

We now consider the case: $h=1,\ \mathcal{M}=I_h=1,\ L=\mathbb{Z}^{(1,g)}\cong\mathbb{Z}^g$. We define the theta series $\theta^{(g)}(\Omega)$ by

$$\theta^{(g)}(\Omega) = \sum_{\lambda\in L} e^{2\pi i\,\sigma(\lambda\Omega\,{}^t\lambda)} = \theta_0(\Omega,0), \quad \Omega\in\mathbb{H}_g. \tag{8.5}$$

Let
$$\Gamma_0^{(g)}(4)=\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\Big|\, C \equiv 0 \ (\mathrm{mod}\, 4) \right\}$$
be the congruence subgroup of Γ_g. We define the automorphic factor $j : \Gamma_0^{(g)}(4)\times \mathbb{H}_g \longrightarrow \mathbb{C}^\times$ by
$$j(\gamma,\Omega)=\frac{\theta^{(g)}(\gamma\cdot\Omega)}{\theta^{(g)}(\Omega)},\quad \gamma\in\Gamma_0^{(g)}(4),\ \Omega\in\mathbb{H}_g.$$
Thus one obtains the relation
$$j(\gamma,\Omega)^2=\varepsilon(\gamma)\,\det(C\Omega+D),\quad \varepsilon(\gamma)^2=1,$$
for any $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_0^{(g)}(4)$.

Kohnen [76] introduced the so-called *Kohnen plus space* $M^+_{k-\frac12}\big(\Gamma_0^{(g)}(4)\big)$ consisting of holomorphic functions satisfying the following conditions $(K1)$ and $(K2)$:

(K1) $\quad f(\gamma\cdot\Omega)=j(\gamma,\Omega)^{2k-1}f(\Omega)\quad$ for all $\gamma\in\Gamma_0^{(g)}(4)$;

(K2) f has the Fourier expansion
$$f(\Omega)=\sum_{T\geq 0} a(T)\,e^{2\pi i\,\sigma(T\Omega)},$$
where T runs over the set of semi-positive half-integral symmetric matrices of degree g such that $a(T)=0$ unless $T\equiv -\mu\,{}^t\mu \bmod 4S_g^*(\mathbb{Z})$ for some $\mu\in\mathbb{Z}^{(g,1)}$. Here we put
$$S_g^*(\mathbb{Z})=\left\{ T\in\mathbb{R}^{(g,g)}\mid T={}^tT,\ \sigma(TS)\in\mathbb{Z}\ \text{ for all } S={}^tS\in\mathbb{Z}^{(g,g)} \right\}.$$

For a Jacobi form $\phi\in J_{k,1}(\Gamma_g)$, according to Formula (8.4), one has

(8.6) $$\phi(\Omega,Z)=\sum_{\gamma\in L/2L} f_\gamma(\Omega)\,\theta_\gamma(\Omega,Z),\quad \Omega\in\mathbb{H}_g,\ Z\in\mathbb{C}^{(h,g)}.$$

Now we put
$$f_\phi(\Omega):=\sum_{\gamma\in L/2L} f_\gamma(4\Omega),\quad \Omega\in\mathbb{H}_g.$$
Then $f_\phi\in M^+_{k-\frac12}\big(\Gamma_0^{(g)}(4)\big)$.

Theorem 8.1. ***(Kohnen-Zagier (g=1), Ibukiyama*** *($g>1$)) Suppose k is an even positive integer. Then there exists the isomorphism given by*
$$J_{k,1}(\Gamma_g)\cong M^+_{k-\frac12}\big(\Gamma_0^{(g)}(4)\big),\qquad \phi\mapsto f_\phi.$$
Moreover the isomorphism is compatible with the action of Hecke operators.

For a positive integer $k\in\mathbb{Z}^+$, H. Maass [93, 94, 95] introduced the so-called *Maass space* $M_k^*(\Gamma_2)$ consisting of all Siegel modular forms $F(\Omega)=\sum_{g\geq 0} a_F(T)\,e^{2\pi i\,\sigma(T\Omega)}$ on $\mathbb{H}_2$ of weight k satisfying the following condition

(8.7) $$a_F(T)=\sum_{d|(n,r,m),\,d>0} d^{k-1}\,a_F\begin{pmatrix} \frac{dm}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix}$$

for all $T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \geq 0$ with $n, r, m \in \mathbb{Z}$. For $F \in M_k(\Gamma_2)$, we let

$$F(\Omega) = \sum_{m \geq 0} \phi_m(\tau, z)\, e^{2\pi i m \tau'}, \qquad \Omega = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{H}_2$$

be the Fourier-Jacobi expansion of F. Then for any nonnegative integer m, we obtain the linear map

$$\rho_m : M_k(\Gamma_2) \longrightarrow J_{k,m}(\Gamma_1), \qquad F \mapsto \phi_m.$$

We observe that ρ_0 is nothing but the Siegel Φ-operator. Maass [93, 94, 95] showed that for k even, there exists a nontrivial map $V : J_{k,1}(\Gamma_1) \longrightarrow M_k(\Gamma_2)$ such that $\rho_1 \circ V$ is the identity. More precisely, we let $\phi \in J_{k,1}(\Gamma_1)$ be a Jacobi form with Fourier coefficients $c(n,r)\,(n, r \in \mathbb{Z},\ r^2 \leq 4n)$ and define for any nonnegative integer $m \geq 0$

$$(8.8) \qquad \big(V_m\phi\big)(\tau, z) = \sum_{n,r \in \mathbb{Z},\, r^2 \leq 4mn} \left(\sum_{d|(n,r,m)} d^{k-1}\, c\left(\frac{mn}{d^2}, \frac{r}{d}\right) \right) e^{2\pi i (n\tau + rz)}.$$

It is easy to see that $V_1\phi = \phi$ and $V_m\phi \in J_{k,m}(\Gamma_1)$. We define

$$(8.9) \qquad (V\phi)(\Omega) = \sum_{m \geq 0} (V_m\phi)(\tau, z)\, e^{2\pi i m \tau'}, \quad \Omega = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{H}_2.$$

We denote by $T_n\ (\, n \in \mathbb{Z}^+)$ the usual Hecke operators on $M_k(\Gamma_2)$ resp. $S_k(\Gamma_2)$. For instance, if p is a prime, $T_p = T(p)$ and $T_{p^2} = T_1(p^2)$. We denote by $T_{J,n}\,(m \in \mathbb{Z}^+)$ the Hecke operators on $J_{k,m}(\Gamma_1)$ resp. $J^{\text{cusp}}_{k,m}(\Gamma_1)$ (cf. [29]).

Theorem 8.2. ***(Maass*** [92, 93, 94]***, Eichler-Zagier*** [29]***, Theorem 6.3)*** *Suppose k is an even positive integer. Then the map $\phi \mapsto V\phi$ gives an isomorphism of $J_{k,m}(\Gamma_1)$ onto $M_k^*(\Gamma_2)$ which sends cusp Jacobi forms to cusp forms and is compatible with the action of Hecke operators. If p is a prime, one has*

$$T_p \circ V = V \circ \big(T_{J,p} + p^{k-2}(p+1)\big)$$

and

$$T_{p^2} \circ V = V \circ \big(T_{J,p}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2}\big).$$

In Summary, we have the following isomorphisms

$$(8.10) \qquad \begin{array}{ccccccc} M_k^*(\Gamma_2) & \cong & J_{k,m}(\Gamma_1) & \cong & M^+_{k-\frac{1}{2}}\big(\Gamma_0^{(1)}(4)\big) & \cong & M_{2k-2}(\Gamma_1), \\ V_\phi & \longleftarrow & \phi & \longrightarrow & f_\phi & & \end{array}$$

where the last isomorphism is the Shimura correspondence. All the above isomorphisms are compatible with the action of Hecke operators.

In 1978, providing some evidences, Kurokawa and Saito conjectured that there is a one-to-one correspondence between Hecke eigenforms in $S_{2k-2}(\Gamma_1)$ and Hecke eigenforms in $M_k(\Gamma_2)$ satisfying natural identity between their spinor zeta functions. This was solved mainly by Maass and then completely solved by Andrianov [1] and Zagier [164].

Theorem 8.3. *Suppose k is an even positive integer and let $F \in M_k^*(\Gamma_2)$ be a nonzero Hecke eigenform. Then there exists a unique normalized Hecke eigenform f in $M_{2k-2}(\Gamma_1)$ such that*

$$Z_F(s) = \zeta(s-k+)\,\zeta(s-k+2)L(f,s), \tag{8.11}$$

where $L(f,s)$ is the Hecke L-function attached to f.

F is called the *Saito-Kurokawa lift* of f. Theorem 8.3 implies that $Z_F(s)$ has a pole at $s=k$ if F is an eigenform in $M_k^*(\Gamma_2)$. If $F \in S_k(\Gamma_2)$ is a Hecke eigenform, it was proved by Andrianov [2] that $Z_F(s)$ has an analytic continuation to the whole complex plane which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at $s=k$ if k is even. Moreover the global function

$$Z_F^*(s) := (2\pi)^{-s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

is $(-1)^k$-invariant under $s \mapsto 2k-2-s$. It was proved that Evdokimov and Oda that $Z_F(s)$ is holomorphic if and only if F is contained in the orthogonal complement of $M_k^*(\Gamma_2)$ in $M_k(\Gamma_2)$. We remark that $M_k(\Gamma_2) = \mathbb{C}\,G_k \oplus S_k^*(\Gamma_2)$, where G_k is the Siegel Eisenstein series of degree 2 (cf. (6.14)) and $S_k^*(\Gamma_2) = S_k(\Gamma_2) \cap M_k^*(\Gamma_2)$.

Around 1996, Duke and Imamoḡlu [26] conjectured a generalization of Theorem 7.3. More precisely, they formulated the conjecture that if f is a normalized Hecke eigenform in $S_{2k}(\Gamma_1)\,(k \in \mathbb{Z}^+)$ and n is a positive integer with $n \equiv k \,(\mathrm{mod}\ 2)$, then there exists a Hecke eigenform F in $S_{k+n}(\Gamma_{2n})$ such that the standard zeta function $D_F(s)$ of F equals

$$\zeta(s)\sum_{j=1}^{2n} L(f, s+k+n-j), \tag{8.12}$$

where $L(f,s)$ is the Hecke L-function of f. Later some evidence for this conjecture was given by Breulmann and Kuss [18]. In 1999, Ikeda [66] proved that the conjecture of Duke and Imamoḡlu is true. Such a Hecke eigenform F in $S_{k+n}(\Gamma_{2n})$ is called the *Duke-Imamoḡlu-Ikeda lift* of a normalized Hecke eigenform f in $S_{2k}(\Gamma_1)$.

Now we describe the work of Tamotsu Ikeda roughly. First we introduce some notations and recall some definitions. A symmetric square matrix A with entries a_{ij} in the quotient field of an integral domain R will be said to be *half integral* if $a_{ii} \in R$ for all i and $2a_{ij} \in R$ for all i,j with $i \neq j$. We denote by $\mathcal{S}_n(R)$ the set of all such symmetric half integral matrices of degree n. For a rational, half integral symmetric, non-degenerate matrix $T \in \mathcal{S}_{2n}(\mathbb{Q})$, we denote by

$$D_T := (-1)^n \det(2T)$$

the discriminant of T. We write

$$D_T = D_{T,0} f_T^2$$

with $D_{T,0}$ the corresponding fundamental discriminant and $f_T \in \mathbb{Z}^+$.

Fix a prime p. Let T be a non-degenerate matrix in $\mathcal{S}_{2n}(\mathbb{Z}_p)$. Then the local singular series of T at p is defined as

$$b_p(T;s) := \sum_R \nu_p(R)^{-s}\, e_p(\sigma(TR)), \qquad s \in \mathbb{C},$$

where R runs over all symmetric $2n \times 2n$ matrices with entries in $\mathbb{Q}_p/\mathbb{Z}_p$ and $\nu_p(R)$ is a power of p equal to the product of denominators of elementary divisors of R. Furthermore, for $x \in \mathbb{Q}_p$ we have put $e_p(x) = e^{2\pi i x'}$, where x' denotes the fractional part of x.

As is well known, $b_p(T;s)$ is a product of two polynomials in p^{-s} with coefficients in $\mathbb{Z}$. More precisely, we put

$$\gamma_p(T;X) := (1-X)\big(1-\xi_p(T)p^nX\big)^{-1}\prod_{j=1}^{n}\big(1-p^{2j}X^2\big),$$

where

$$\xi_p(T) := \chi_p\big((-1)^n \det T\big)$$

and for $a \in \mathbb{Q}_p^*$, $\chi_p(a)$ is defined by

$$\chi_p(a) = \begin{cases} \ \ 1 & \text{if } \ \mathbb{Q}_p(\sqrt{a}) = \mathbb{Q}_p, \\ -1 & \text{if } \ \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is unramified}, \\ \ \ 0 & \text{if } \ \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is ramified}. \end{cases}$$

Then we have

$$b_p(T;s) = \gamma_p(T;p^{-s})\, F_p(T;p^{-s}),$$

where $F_p(T;X)$ is a certain polynomial in $\mathbb{Z}[X]$ with constant term 1. A fundamental result of Katsurada [71] states that the Laurent polynomial

$$\widetilde{F}_p(T;X) := X^{-\mathrm{ord}_p f_T} F_p(T;p^{-n-1/2}X)$$

is symmetric, i.e.,

$$\widetilde{F}_p(T;X) = \widetilde{F}_p\big(T;X^{-1}\big),$$

where ord_p denotes the usual p-adic valuation on $\mathbb{Q}$. If p does not divide f_T, then $F_p(T;X) = \widetilde{F}_p(T;X) = 1$. We denote by $V = \big(\mathbb{F}_p^{2n}, q\big)$ the quadratic space over $\mathbb{F}_p$, where q is the quadratic form obtained from the quadratic form $x \mapsto T[x]$ $(x \in \mathbb{Z}_p^{2n})$ by reducing modulo p. We let $\langle\ ,\ \rangle$ be the associated bilinear form on $\mathbb{F}_p^{2n}$ given by

$$\langle x,y\rangle := q(x+y)-q(x)-q(y), \quad x,y \in \mathbb{F}_p^{2n}$$

and let

$$R(V) := \big\{x \in \mathbb{F}_p^{2n} \mid \langle x,y\rangle = 0 \ \text{ for all } y \in \mathbb{F}_p^{2n},\ q(x)=0\,\big\}$$

be the radical of V. We put $s_p := s_p(T) = \dim R(V)$ and denote by W an orthogonal complementary subspace of $R(V)$.

Following [72], one defines a polynomial $H_{n,p}(T;X)$ by

$$H_{n,p}(T;X) = \begin{cases} 1 & \text{if } s_p = 0, \\ \prod_{j=1}^{[(s_p-1)/2]} \left(1 - p^{2j-1}X^2\right), & \text{if } s_p > 0,\ s_p \text{ odd}, \\ \left(1 + \lambda_p(T) p^{(s_p-1)/2} X\right) \prod_{j=1}^{[(s_p-1)/2]} \left(1 - p^{2j-1}X^2\right), & \text{if } s_p > 0,\ s_p \text{ even}, \end{cases}$$

where $[x]$ denotes the Gauss bracket of a real number x, and for s_p *even* we have put

$$\lambda_p(T) := \begin{cases} 1 & \text{if } W \text{ is a hyperbolic subspace or } s_p = 2n, \\ -1 & \text{otherwise.} \end{cases}$$

Following [77], for $\mu \in \mathbb{Z},\ \mu \geq 0$, we define $\rho_T(p^\mu)$ by

$$\sum_{\mu \geq 0} \rho_T(p^\mu) X^\mu := \begin{cases} (1 - X^2)\, H_{n,p}(T;X) & \text{if } p \mid f_T, \\ 1 & \text{otherwise.} \end{cases}$$

We extend the function ρ_T multiplicatively to $\mathbb{Z}^+$ by defining

$$\sum_{a \geq 1} \rho_T(a)\, a^{-s} := \prod_{p \mid f_T} \left(\left(1 - p^{-2s}\right) H_{n,p}(T; p^{-s}) \right).$$

Let

$$\mathcal{D}(T) := GL_{2n}(\mathbb{Z}) \backslash \left\{ G \in M_{2n}(\mathbb{Z}) \cap GL_{2n}(\mathbb{Q}) \mid T[G^{-1}] \text{ half integral} \right\},$$

where $GL_{2n}(\mathbb{Z})$ acts by left multiplication. We see easily that $\mathcal{D}(T)$ is finite. For $a \in \mathbb{Z}^+$ with $a \mid f_T$, we define

$$\phi(a;T) := \sqrt{a} \sum_{d^2 \mid a} \sum_{G \in \mathcal{D}(T),\, |\det(G)| = d} \rho_{T[G^{-1}]} \left(\frac{a}{d^2} \right).$$

We observe that $\phi(a;T) \in \mathbb{Z}$ for all a.

Let f be a normalized Hecke eigenform in $S_{2k}(\Gamma_1)$. For a prime p, we let $\lambda(p)$ and α_p be the p-th Fourier coefficient and the Satake p-parameter of f respectively. Therefore one has

$$1 - \lambda(p)X + p^{2k-1}X^2 = \left(1 - p^{k-1/2}\alpha_p X\right)\left(1 - p^{k-1/2}\alpha_p^{-1} X\right).$$

Let

$$g(\tau) = \sum_{m \geq 1,\, (-1)^k m \equiv 0,1 \pmod 4} c(m)\, e^{2\pi i m \tau}, \qquad \tau \in \mathbb{H}_1$$

be a Hecke eigenform in $S^+_{k+\frac{1}{2}}\left(\Gamma_0^{(1)}(4)\right)$ which corresponds to f under the Shimura isomorphism (8.10). Now we assume that n is a positive integer satisfying the condition $n \equiv k \pmod 2$. For a rational, half integral symmetric positive definite matrix T of degree $2n$, we define

$$a_f(T) := c(|D_{T,0}|)\, f_T^{k-\frac{1}{2}} \prod_{p \mid f_T} \tilde{F}_p(T; \alpha_p).$$

We consider the function $F(\Omega)$ defined by

$$F(\Omega) = \sum_{T > 0} a_f(T)\, e^{2\pi i\, \sigma(T\Omega)},$$

where T runs over all rational, half integral symmetric positive definite matrices of degree $2n$. Ikeda [66] proved that $F(\Omega)$ is a cuspidal Siegel-Hecke eigenform in $S_{k+n}(\Gamma_{2n})$ and the standard zeta function $D_F(s)$ of F is given by the formula (8.12). Therefore we have the mapping

$$I_{k,n} : S^+_{k+\frac{1}{2}}\big(\Gamma_0^{(1)}(4)\big) \longrightarrow S_{k+n}(\Gamma_{2n}) \tag{8.13}$$

defined by

$$g(\tau) = \sum_{(-1)^k m\equiv 0,1 (\mathrm{mod}\, 4)} c(m)e^{2\pi i m\tau} \longmapsto F(\Omega) = \sum_{T>0} A(T)\, e^{2\pi i\, \sigma(T\Omega)},$$

where T runs over all rational, half integral symmetric positive definite matrices of degree $2n$ and

$$A(T) = c(|D_{T,0}|)\, f_T^{k-\frac{1}{2}} \prod_{p|f_T} \tilde{F}_p(T;\alpha_p).$$

The mapping $I_{k,n}$ is called the *Ikeda's lift map.* Kohnen [77] showed the following identity

$$a_f(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)\, c(|D_T|/a^2).$$

Kohnen and Kojima [78] characterized the image $S^*_{k+n}(\Gamma_{2n})$ of the Ikeda's lift map $I_{k,n}$ as follows:

Theorem 8.4. ***(Kohnen-Kojima*** [78]*) Suppose that $n \equiv 0, 1$ (mod 4) and let $k \in \mathbb{Z}^+$ with $n \equiv k$ (mod 2). Let $F \in S_{k+n}(\Gamma_{2n})$ with Fourier coefficient $A(T)$. Then the following statements are equivalent:*

(a) $F \in S^*_{k+n}(\Gamma_{2n})$;

(b) there exist complex numbers $c(m)$ (with $m \in \mathbb{Z}^+$, and $(-1)^k m \equiv 0, 1$ (mod 4)) such that

$$A(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)\, c(|D_T|/a^2)$$

for all T.

They called the image of $I_{k,n}$ in $S_{k+n}(\Gamma_{2n})$ the *Maass space.* If $n = 1$, $M^*_k(\Gamma_2)$ coincides with the image of $I_{k,1}$. Thus this generalizes the original Maass space. Breulmann and Kuss [18] dealt with the special case of the lift map $I_{6,2} : S_{12}(\Gamma_1)(\cong S^+_{13/2}) \longrightarrow S_8(\Gamma_4)$. In the article [17], starting with the Leech lattice Λ, the authors constructed a nonzero Siegel cusp form of degree 12 and weight 12 which is the image of a cusp form $\Delta \in S_{12}(\Gamma_1)$ under the Ikeda lift map $I_{6,6}$. Here Δ is the cusp form in $S_{12}(\Gamma_1)$ defined by

$$\Delta(\tau) = (2\pi)^{12}\, q \prod_{n=1}^{\infty}(1-q^n)^{24}, \quad \tau \in \mathbb{H}_1,\ q = e^{2\pi i\tau}.$$

It is known that there exist 24 Niemeier lattices of rank 24, say, $L_1, \cdots, L_{24}$. The theta series

$$\theta_{L_i}(\Omega) = \sum_{G\in\mathbb{Z}^{(24,12)}} e^{2\pi i\, \sigma(L_i[G]\Omega)}, \quad \Omega \in \mathbb{H}_{12}, \quad i = 1, \cdots, 24$$

generate a subspace V_* of $M_{12}(\Gamma_{12})$. These θ_{L_i} $(1 \leq i \leq 24)$ are linearly independent. It can be seen that the intersection $V_* \cap S_{12}(\Gamma_{12})$ is one dimensional. This nontrivial cusp form in $V_* \cap S_{12}(\Gamma_{12})$ up to constant is just the Siegel modular form constructed by them. Under the assumption $n + r \equiv k \,(\mathrm{mod}\, 2)$ with $k, n, r \in \mathbb{Z}^+$, using the lift map $I_{k,n+r} : S^+_{k+\frac{1}{2}} \longrightarrow S_{k+n+r}(\Gamma_{2n+2r})$, recently Ikeda [67] constructed the following map

$$J_{k,n,r} : S^+_{k+\frac{1}{2}} \times S_{k+n+r}(\Gamma_r) \longrightarrow S_{k+n+r}(\Gamma_{2n+r}) \tag{8.14}$$

defined by

$$J_{k,n,r}(h, G)(\Omega) := \int_{\Gamma_r \backslash \mathbb{H}_r} I_{k,n+r}(h) \left(\begin{pmatrix} \Omega & 0 \\ 0 & \tau \end{pmatrix} \right) \overline{G^c(\tau)} \left(\det \operatorname{Im} \tau \right)^{k+n-1} d\tau,$$

where $h \in S^+_{k+\frac{1}{2}}$, $G \in S_{k+n+r}(\Gamma_r)$, $\Omega \in \mathbb{H}_{2n+r}$, $\tau \in \mathbb{H}_r$, $G^c(\tau) = \overline{G(-\overline{\tau})}$ and $(\det \operatorname{Im} \tau)^{-(r+1)} d\tau$ is an invariant volume element (cf. §2 (2.3)). He proved that the standard zeta function $D_{J_{k,n,r}(h,G)}(s)$ of $J_{k,n,r}(h, G)$ is equal to

$$D_{J_{k,n,r}(h,G)}(s) = D_G(s) \prod_{j=1}^{n} L(f, s + k + n - j),$$

where f is the Hecke eigenform in $S_{2k}(\Gamma_1)$ corresponding to $h \in S^+_{k+\frac{1}{2}}$ under the Shimura correspondence.

Question : Can you describe a geometric interpretation of the Duke-Imamoğlu-Ikeda lift or the map $J_{k,n,r}$?

9. Holomorphic Differential Forms on Siegel Space

In this section, we describe the relationship between Siegel modular forms and holomorphic differential forms on the Siegel space. We also discuss the Hodge bundle. First of all we need to know the theory of toroidal compactifications of the Siegel space. We refer to [5, 107, 140] for the detail on toroidal compactifications of the Siegel space.

For a neat arithmetic subgroup Γ, e.g., $\Gamma = \Gamma_g(n)$ with $n \geq 3$, we can obtain a smooth projective toroidal compactification of $\Gamma \backslash \mathbb{D}_g$. The theory of toroidal compactifications of bounded symmetric domains was developed by Mumford's school (cf. [5] and [107]). We set

$$\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g \quad \text{and} \quad \mathcal{A}_g^* := \Gamma_g \backslash \mathbb{H}_g^* = \bigcup_{0 \leq i \leq g} \Gamma_i \backslash \mathbb{H}_i \quad \text{(disjoint union)}.$$

I. Satake [117] showed that $\mathcal{A}_g^*$ is a normal analytic space and W. Baily [6] proved that $\mathcal{A}_g^*$ is a projective variety. Let $\tilde{\mathcal{A}}_g$ be a toroidal compactification of $\mathcal{A}_g$. Then the boundary $\tilde{\mathcal{A}}_g - \mathcal{A}_g$ is a divisor with normal crossings and one has a universal semi-abelian variety over $\tilde{\mathcal{A}}_g$ in the orbifold. We refer to [59] for the geometry of $\mathcal{A}_g$.

Let θ be the second symmetric power of the standard representation of $GL(g, \mathbb{C})$. For brevity we set $N = \frac{1}{2} g(g+1)$. For an integer p with $0 \leq p \leq N$, we denote by $\theta^{[p]}$ the p-th exterior power of θ. For any integer q with $0 \leq q \leq N$, we let $\Omega^q(\mathbb{H}_g)^{\Gamma_g}$ be the vector space of all Γ_g-invariant holomorphic q-forms on $\mathbb{H}_g$. Then we obtain an isomorphism

$$\Omega^q(\mathbb{H}_g)^{\Gamma_g} \longrightarrow M_{\theta^{[q]}}(\Gamma_g).$$

Theorem 9.1. ***(Weissauer*** [143]***)*** *For an integer α with $0 \leq \alpha \leq g$, we let ρ_α be the irreducible representation of $GL(g, \mathbb{C})$ with the highest weight*

$$(g+1, \cdots, g+1, g-\alpha, \cdots, g-\alpha)$$

such that corank$(\rho_\alpha) = \alpha$ for $1 \leq \alpha \leq g$. If $\alpha = -1$, we let $\rho_\alpha = (g+1, \cdots, g+1)$. Then

$$\Omega^q(\mathbb{H}_g)^{\Gamma_g} = \begin{cases} M_{\rho_\alpha}(\Gamma_g) & \text{if } q = \frac{g(g+1)}{2} - \frac{\alpha(\alpha+1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Remark. If $2\alpha > g$, then any $f \in M_{\rho_\alpha}(\Gamma_g)$ is singular (cf. Theorem 5.4). Thus if $q < \frac{g(3g+2)}{8}$, then any Γ_g-invariant holomorphic q-form on $\mathbb{H}_g$ can be expressed in terms of vector valued theta series with harmonic coefficients. It can be shown with a suitable modification that the just mentioned statement holds for a sufficiently small congruence subgroup of Γ_g.

Thus the natural question is to ask how to determine the Γ_g-invariant holomorphic p-forms on $\mathbb{H}_g$ for the nonsingular range $\dfrac{g(3g+2)}{8} \leq p \leq \dfrac{g(g+1)}{2}$. Weissauer [144] answered the above question for $g = 2$. For $g > 2$, the above question is still open. It is well known that the vector space of vector valued modular forms of type ρ is finite dimensional. The computation or the estimate of the dimension of $\Omega^p(\mathbb{H}_g)^{\Gamma_g}$ is interesting because its dimension is finite even though the quotient space $\mathcal{A}_g$ is noncompact.

Example 1. Let

$$\varphi = \sum_{i \leq j} f_{ij}(\Omega)\, d\omega_{ij} \tag{9.1}$$

be a Γ_g-invariant holomorphic 1-form on $\mathbb{H}_g$. We put

$$f(\Omega) = \big(f_{ij}(\Omega)\big) \text{ with } f_{ij} = f_{ji} \quad \text{and} \quad d\Omega = (d\omega_{ij}).$$

Then f is a matrix valued function on $\mathbb{H}_g$ satisfying the condition

$$f(\gamma \cdot \Omega) = (C\Omega + D) f(\Omega)\, {}^t(C\Omega + D) \quad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \text{ and } \Omega \in \mathbb{H}_g.$$

This implies that f is a Siegel modular form in $M_\theta(\Gamma_g)$, where θ is the irreducible representation of $GL(g, \mathbb{C})$ on $T_g = \mathrm{Symm}^2(\mathbb{C}^g)$ defined by

$$\theta(h)v = h\, v\, {}^t h, \quad h \in GL(g, \mathbb{C}),\ v \in T_g.$$

We observe that (9.6) can be expressed as $\varphi = \sigma(f\, d\Omega)$.

Example 2. Let

$$\omega_0 = d\omega_{11} \wedge d\omega_{12} \wedge \cdots \wedge d\omega_{gg}$$

be a holomorphic N-form on $\mathbb{H}_g$. If $\omega = f(\Omega)\,\omega_0$ is Γ_g-invariant, it is easily seen that

$$f(\gamma\cdot\Omega)=\det(C\Omega+D)^{g+1}f(\Omega)\quad \text{for all } \gamma=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\Gamma_g \text{ and } \Omega\in\mathbb{H}_g.$$

Thus $f\in M_{g+1}(\Gamma_g)$. It was shown by Freitag [34] that ω can be extended to a holomorphic N-form on $\tilde{\mathcal{A}}_g$ if and only if f is a cusp form in $S_{g+1}(\Gamma_g)$. Indeed, the mapping

$$S_{g+1}(\Gamma_g)\longrightarrow \Omega^N\big(\tilde{\mathcal{A}}_g\big)=H^0\big(\tilde{\mathcal{A}}_g,\Omega^N\big),\qquad f\mapsto f\,\omega_0$$

is an isomorphism. Let $\omega_k=F(\Omega)\,\omega_0^{\otimes k}$ be a Γ_g-invariant holomorphic form on $\mathbb{H}_g$ of degree kN. Then $F\in M_{k(g+1)}(\Gamma_g)$.

Example 3. We set

$$\eta_{ab}=\epsilon_{ab}\bigwedge_{\substack{1\le\mu\le\nu\le g\\ (\mu,\nu)\neq(a,b)}} d\omega_{\mu\nu},\quad 1\le a\le b\le g,$$

where the signs ϵ_{ab} are determined by the relations $\epsilon_{ab}\,\eta_{ab}\wedge d\omega_{ab}=\omega_0$. We assume that

$$\eta_*=\sum_{1\le a\le b\le g} F_{ab}\,\eta_{ab}$$

is a Γ_g-invariant holomorphic $(N-1)$-form on $\mathbb{H}_g$. Then the matrix valued function $F=\big(\epsilon_{ab}\,F_{ab}\big)$ with $\epsilon_{ab}=\epsilon_{ba}$ and $F_{ab}=F_{ba}$ is an element of $M_\tau(\Gamma_g)$, where τ is the irreducible representation of $GL(g,\mathbb{C})$ on T_g defined by

$$\tau(h)v=(\det h)^{g+1}\,{}^t h^{-1}vh^{-1},\quad h\in GL(g,\mathbb{C}),\ v\in T_g.$$

We will mention the results due to Weissauer [144]. We let Γ be a congruence subgroup of Γ_2. According to Theorem 9.1, Γ-invariant holomorphic forms in $\Omega^2(\mathbb{H}_2)^\Gamma$ are corresponded to modular forms of type (3,1). We note that these invariant holomorphic 2-forms are contained in the *nonsingular range*. And if these modular forms are not cusp forms, they are mapped under the Siegel Φ-operator to cusp forms of weight 3 with respect to some congruence subgroup (dependent on Γ) of the elliptic modular group. Since there are finitely many cusps, it is easy to deal with these modular forms in the adelic version. Observing these facts, he showed that any 2-holomorphic form on $\Gamma\backslash\mathbb{H}_2$ can be expressed in terms of theta series with harmonic coefficients associated to binary positive definite quadratic forms. Moreover he showed that $H^2(\Gamma\backslash\mathbb{H}_2,\mathbb{C})$ has a pure Hodge structure and that the Tate conjecture holds for a suitable compactification of $\Gamma\backslash\mathbb{H}_2$. If $g\ge 3$, for a congruence subgroup Γ of Γ_g it is difficult to compute the cohomology groups $H^*(\Gamma\backslash\mathbb{H}_g,\mathbb{C})$ because $\Gamma\backslash\mathbb{H}_g$ is noncompact and highly singular. Therefore in order to study their structure, it is natural to ask if they have pure Hodge structures or mixed Hodge structures.

We now discuss the Hodge bundle on the Siegel modular variety $\mathcal{A}_g$. For simplicity we take $\Gamma=\Gamma_g(n)$ with $n\ge 3$ instead of Γ_g. We recall that $\Gamma_g(n)$ is a congruence subgroup of Γ_g consisting of matrices $M\in\Gamma_g$ such that $M\equiv I_{2g}\,(\mathrm{mod}\ n)$. Let

$$\mathfrak{X}_g(n):=\Gamma_g(n)\ltimes\mathbb{Z}^{2g}\backslash\mathbb{H}_g\times\mathbb{C}^g$$

be a family of abelian varieties of dimension g over $\mathcal{A}_g(n) := \Gamma_g(n)\backslash\mathbb{H}_g$. We recall that $\Gamma_g(n) \ltimes \mathbb{Z}^{2g}$ acts on $\mathbb{H}_g \times \mathbb{C}^g$ freely by

$$\big(\gamma, (\lambda, \mu)\big) \cdot (\Omega, Z) = \big(\gamma \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}\big),$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n),\ \lambda, \mu \in \mathbb{Z}^g,\ \Omega \in \mathbb{H}_g$ and $Z \in \mathbb{C}^g$. If we insist on using Γ_g, we need to work with orbifolds or stacks to have a universal family

$$\mathfrak{X}_g := \mathfrak{X}_g(n)/Sp(g, \mathbb{Z}/n\mathbb{Z})$$

available. We observe that $\Gamma_g(n)$ acts on $\mathbb{H}_g$ freely. Therefore we obtain a vector bundle $\mathbb{E} = \mathbb{E}_g$ over $\mathcal{A}_g(n)$ of rank g

$$\mathbb{E} = \mathbb{E}_g := \Gamma_g(n)\backslash\big(\mathbb{H}_g \times \mathbb{C}^g\big).$$

This bundle $\mathbb{E}$ is called the *Hodge bundle* over $\mathcal{A}_g(n)$. The finite group $Sp(g, \mathbb{Z}/n\mathbb{Z})$ acts on $\mathbb{E}$ and a $Sp(g, \mathbb{Z}/n\mathbb{Z})$-invariant section of $(\det \mathbb{E})^{\otimes k}$ with a positive integer k comes from a Siegel modular form of weight k in $M_k(\Gamma_g)$. The canonical line bundle $\kappa_g(n)$ of $\mathcal{A}_g(n)$ is isomorphic to $(\det \mathbb{E})^{\otimes(g+1)}$. A holomorphic section of $\kappa_g(n)$ corresponds to a Siegel modular form in $M_{g+1}(\Gamma_g(n))$ (cf. Example 2). We note that the sheaf $\Omega^1_{\mathcal{A}_g(n)}$ of holomorphic 1-forms on $\mathcal{A}_g(n)$ is isomorphic to $\mathrm{Symm}^2(\mathbb{E})$. This sheaf can be extended over a toroidal compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ to an isomorphism

$$\Omega^1_{\tilde{\mathcal{A}}_g}(\log D) \cong \mathrm{Symm}^2(\mathbb{E}),$$

where the boundary $D = \tilde{\mathcal{A}}_g - \mathcal{A}_g$ is the divisor with normal crossings. Similarly to each finite dimensional representation (ρ, V_ρ) of $GL(g, \mathbb{C})$, we may associate the vector bundle

$$\mathbb{E}_\rho := \Gamma_g(n)\backslash\big(\mathbb{H}_g \times V_\rho\big)$$

by identifying (Ω, v) with $(\gamma \cdot \Omega, \rho(C\Omega + D)v)$, where $\Omega \in \mathbb{H}_g,\ v \in V_\rho$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)$. Obviously $\mathbb{E}_\rho$ is a holomorphic vector bundle over $\mathcal{A}_g(n)$ of rank $\dim V_\rho$.

10. **Subvarieties of the Siegel Modular Variety**

Here we assume that the ground field is the complex number field $\mathbb{C}$.

Definition 9.1. A nonsingular variety X is said to be *rational* if X is birational to a projective space $\mathbb{P}^n(\mathbb{C})$ for some integer n. A nonsingular variety X is said to be *stably rational* if $X \times \mathbb{P}^k(\mathbb{C})$ is birational to $\mathbb{P}^N(\mathbb{C})$ for certain nonnegative integers k and N. A nonsingular variety X is called *unirational* if there exists a dominant rational map $\varphi : \mathbb{P}^n(\mathbb{C}) \longrightarrow X$ for a certain positive integer n, equivalently if the function field $\mathbb{C}(X)$ of X can be embedded in a purely transcendental extension $\mathbb{C}(z_1, \cdots, z_n)$ of $\mathbb{C}$.

Remarks 9.2. (1) It is easy to see that the rationality implies the stably rationality and that the stably rationality implies the unirationality.

(2) If X is a Riemann surface or a complex surface, then the notions of rationality, stably rationality and unirationality are equivalent one another.

(3) Griffiths and Clemens [21] showed that most of cubic threefolds in $\mathbb{P}^4(\mathbb{C})$ are unirational but *not* rational.

The following natural questions arise :

QUESTION 1. Is a stably rational variety *rational*? Indeed, the question was raised by Bogomolov.

QUESTION 2. Is a general hypersurface $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \leq n+1$ *unirational*?

Definition 9.3. Let X be a nonsingular variety of dimension n and let K_X be the canonical divisor of X. For each positive integer $m \in \mathbb{Z}^+$, we define the *m-genus* $P_m(X)$ of X by

$$P_m(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}(mK_X)).$$

The number $p_g(X) := P_1(X)$ is called the *geometric genus* of X. We let

$$N(X) := \left\{ m \in \mathbb{Z}^+ \,|\, P_m(X) \geq 1 \right\}.$$

For the present, we assume that $N(X)$ is nonempty. For each $m \in N(X)$, we let $\{\phi_0, \cdots, \phi_{N_m}\}$ be a basis of the vector space $H^0(X, \mathcal{O}(mK_X))$. Then we have the mapping $\Phi_{mK_X} : X \longrightarrow \mathbb{P}^{N_m}(\mathbb{C})$ by

$$\Phi_{mK_X}(z) := (\phi_0(z) : \cdots : \phi_{N_m}(z)), \quad z \in X.$$

We define the *Kodaira dimension* $\kappa(X)$ of X by

$$\kappa(X) := \max \left\{ \dim_{\mathbb{C}} \Phi_{mK_X}(X) \,|\, m \in N(X) \right\}.$$

If $N(X)$ is empty, we put $\kappa(X) := -\infty$. Obviously $\kappa(X) \leq \dim_{\mathbb{C}} X$. A nonsingular variety X is said to be *of general type* if $\kappa(X) = \dim_{\mathbb{C}} X$. A singular variety Y in general is said to be rational, stably rational, unirational or of general type if any nonsingular model X of Y is rational, stably rational, unirational or of general type respectively. We define

$$P_m(Y) := P_m(X) \quad \text{and} \quad \kappa(Y) := \kappa(X).$$

A variety Y of dimension n is said to be *of logarithmic general type* if there exists a smooth compactification $\tilde{Y}$ of Y such that $D := \tilde{Y} - Y$ is a divisor with normal crossings only and the transcendence degree of the logarithmic canonical ring

$$\oplus_{m=0}^{\infty} H^0(\tilde{Y}, \, m(K_{\tilde{Y}} + [D]))$$

is $n+1$, i.e., the *logarithmic Kodaira dimension* of Y is n. We observe that the notion of being of logarithmic general type is weaker than that of being of general type.

Let $\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$ be the Siegel modular variety of degree g, that is, the moduli space of principally polarized abelian varieties of dimension g. It has been proved that $\mathcal{A}_g$ is of general type for $g \geq 6$. At first Freitag [32] proved this fact when g is a multiple of 24. Tai [134] proved this fact for $g \geq 9$ and Mumford [102] proved this fact for $g \geq 7$. Recently Grushevsky and Lehavi [45] announced that they proved that the Siegel modular variety $\mathcal{A}_6$ of genus 6 is of general type after constructing a series of new effective geometric divisors on

$\mathcal{A}_g$. Before 2005 it had been known that $\mathcal{A}_g$ is of general type for $g \geq 7$. On the other hand, $\mathcal{A}_g$ is known to be unirational for $g \leq 5$: Donagi [25] for $g = 5$, Clemens [20] for $g = 4$ and classical for $g \leq 3$. For $g = 3$, using the moduli theory of curves, Riemann [111], Weber [142] and Frobenius [36] showed that $\mathcal{A}_3(2) := \Gamma_3(2)\backslash\mathbb{H}_3$ is a rational variety and moreover gave 6 generators of the modular function field $K(\Gamma_3(2))$ written explicitly in terms of derivatives of odd theta functions at the origin. So $\mathcal{A}_3$ is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3 : \Gamma_3(2)] = 1,451,520$. Here $\Gamma_3(2)$ denotes the principal congruence subgroup of Γ_3 of level 2. Furthermore it was shown that $\mathcal{A}_3$ is stably rational(cf. [80], [16]). For a positive integer k, we let $\Gamma_g(k)$ be the principal congruence subgroup of Γ_g of level k. Let $\mathcal{A}_g(k)$ be the moduli space of abelian varieties of dimension g with k-level structure. It is classically known that $\mathcal{A}_g(k)$ is of logarithmic general type for $k \geq 3$ (cf. [101]). Wang [141] proved that $\mathcal{A}_2(k)$ is of general type for $k \geq 4$. On the other hand, van der Geer [37] showed that $\mathcal{A}_2(3)$ is rational. The remaining unsolved problems are summarized as follows:

Problem 1. Is $\mathcal{A}_3$ rational?

Problem 2. Are $\mathcal{A}_4,\ \mathcal{A}_5$ stably rational or rational?

Problem 3. What type of varieties are $\mathcal{A}_g(k)$ for $g \geq 3$ and $k \geq 2$?

We already mentioned that $\mathcal{A}_g$ is of general type if $g \geq 6$. It is natural to ask if the subvarieties of $\mathcal{A}_g$ $(g \geq 6)$ are of general type, in particular the subvarieties of $\mathcal{A}_g$ of codimension one. Freitag [35] showed that there exists a certain bound g_0 such that for $g \geq g_0$, each irreducible subvariety of $\mathcal{A}_g$ of codimension one is of general type. Weissauer [145] proved that every irreducible divisor of $\mathcal{A}_g$ is of general type for $g \geq 10$. Moreover he proved that every subvariety of codimension $\leq g - 13$ in $\mathcal{A}_g$ is of general type for $g \geq 13$. We observe that the smallest known codimension for which there exist subvarieties of $\mathcal{A}_g$ for large g which are not of general type is $g - 1$. $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ is a subvariety of $\mathcal{A}_g$ of codimension $g - 1$ which is not of general type.

Remark. Let $\mathcal{M}_g$ be the coarse moduli space of curves of genus g over $\mathbb{C}$. Then $\mathcal{M}_g$ is an analytic subvariety of $\mathcal{A}_g$ of dimension $3g - 3$. It is known that $\mathcal{M}_g$ is unirational for $g \leq 10$. So the Kodaira dimension $\kappa(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $-\infty$ for $g \leq 10$. Harris and Mumford [48] proved that $\mathcal{M}_g$ is of general type for odd g with $g \geq 25$ and $\kappa(\mathcal{M}_{23}) \geq 0$.

11. **Proportionality Theorem**

In this section we describe the proportionality theorem for the Siegel modular variety following the work of Mumford [101]. Historically F. Hirzebruch [55] first described a beautiful proportionality theorem for the case of a *compact* locally symmetric variety in 1956. We shall state his proportionality theorem roughly. Let D be a bounded symmetric domain and let Γ be a discrete torsion-free co-compact group of automorphisms of D. We assume that the quotient space $X_\Gamma := \Gamma\backslash D$ is a *compact* locally symmetric variety. We denote by $\check{D}$ the *compact dual* of D. Hirzebruch [55] proved that the Chern numbers of X_Γ are proportional to the Chern numbers of $\check{D}$, the constant of proportionality being the volume of X_Γ in a natural metric. Mumford [101] generalized Hirzebruch's proportionality theorem to the case of a noncompact arithmetic variety.

Before we describe the proportionality theorem for the Siegel modular variety, first of all we review the compact dual of the Siegel upper half plane $\mathbb{H}_g$. We note that $\mathbb{H}_g$ is biholomorphic to the generalized unit disk $\mathbb{D}_g$ of degree g through the Cayley transform (2.7). We suppose that $\Lambda = (\mathbb{Z}^{2g}, \langle\ ,\ \rangle)$ is a symplectic lattice with a symplectic form $\langle\ ,\ \rangle$. We extend scalars of the lattice Λ to $\mathbb{C}$. Let

$$\mathfrak{Y}_g := \left\{ L \subset \mathbb{C}^{2g} \mid \dim_{\mathbb{C}} L = g, \quad \langle x, y\rangle = 0 \quad \text{for all } x, y \in L \right\}$$

be the complex Lagrangian Grassmannian variety parameterizing totally isotropic subspaces of complex dimension g. For the present time being, for brevity, we put $G = Sp(g,\mathbb{R})$ and $K = U(g)$. The complexification $G_{\mathbb{C}} = Sp(g,\mathbb{C})$ of G acts on $\mathfrak{Y}_g$ transitively. If H is the isotropy subgroup of $G_{\mathbb{C}}$ fixing the first summand $\mathbb{C}^g$, we can identify $\mathfrak{Y}_g$ with the compact homogeneous space $G_{\mathbb{C}}/H$. We let

$$\mathfrak{Y}_g^+ := \left\{ L \in \mathfrak{Y}_g \mid -i\langle x, \bar{x}\rangle > 0 \quad \text{for all } x (\neq 0) \in L \right\}$$

be an open subset of $\mathfrak{Y}_g$. We see that G acts on $\mathfrak{Y}_g^+$ transitively. It can be shown that $\mathfrak{Y}_g^+$ is biholomorphic to $G/K \cong \mathbb{H}_g$. A basis of a lattice $L \in \mathfrak{Y}_g^+$ is given by a unique $2g \times g$ matrix ${}^t(-I_g\ \Omega)$ with $\Omega \in \mathbb{H}_g$. Therefore we can identify L with Ω in $\mathbb{H}_g$. In this way, we embed $\mathbb{H}_g$ into $\mathfrak{Y}_g$ as an open subset of $\mathfrak{Y}_g$. The complex projective variety $\mathfrak{Y}_g$ is called the *compact dual* of $\mathbb{H}_g$.

Let Γ be an arithmetic subgroup of Γ_g. Let E_0 be a G-equivariant holomorphic vector bundle over $\mathbb{H}_g = G/K$ of rank n. Then E_0 is defined by the representation $\tau : K \longrightarrow GL(n,\mathbb{C})$. That is, $E_0 \cong G \times_K \mathbb{C}^n$ is a homogeneous vector bundle over G/K. We naturally obtain a holomorphic vector bundle E over $\mathcal{A}_{g,\Gamma} := \Gamma\backslash G/K$. E is often called an *automorphic* or *arithmetic* vector bundle over $\mathcal{A}_{g,\Gamma}$. Since K is compact, E_0 carries a G-equivariant Hermitian metric h_0 which induces a Hermitian metric h on E. According to Main Theorem in [101], E admits a *unique* extension $\tilde{E}$ to a smooth toroidal compactification $\tilde{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma}$ such that h is a singular Hermitian metric *good* on $\tilde{\mathcal{A}}_{g,\Gamma}$. For the precise definition of a *good metric* on $\mathcal{A}_{g,\Gamma}$ we refer to [101, p. 242]. According to Hirzebruch-Mumford's Proportionality Theorem (cf. [101, p. 262]), there is a natural metric on $G/K = \mathbb{H}_g$ such that the Chern numbers satisfy the following relation

$$c^{\alpha}\big(\tilde{E}\big) = (-1)^{\frac{1}{2}g(g+1)} \operatorname{vol}(\Gamma\backslash\mathbb{H}_g)\, c^{\alpha}\big(\check{E}_0\big) \tag{11.1}$$

for all $\alpha = (\alpha_1, \cdots, \alpha_n)$ with nonnegative integers $\alpha_i\ (1 \le i \le n)$ and $\sum_{i=1}^n \alpha_i = \frac{1}{2}g(g+1)$, where $\check{E}_0$ is the $G_{\mathbb{C}}$-equivariant holomorphic vector bundle on the compact dual $\mathfrak{Y}_g$ of $\mathbb{H}_g$ defined by a certain representation of the stabilizer $\mathrm{Stab}_{G_{\mathbb{C}}}(e)$ of a point e in $\mathfrak{Y}_g$. Here $\operatorname{vol}(\Gamma\backslash\mathbb{H}_g)$ is the volume of $\Gamma\backslash\mathbb{H}_g$ that can be computed (cf. [131]).

Remark 11.1. Goresky and Pardon [41] investigated Chern numbers of an automorphic vector bundle over the Baily-Borel compactification $\overline{X}$ of a Shimura variety X. It is known that $\overline{X}$ is usually a highly singular complex projective variety. They also described the close relationship between the topology of X and the characteristic classes of the unique extension $\widetilde{TX}$ of the tangent bundle TX of X to a smooth toroidal compactification $\tilde{X}$ of X.

12. **Motives and Siegel Modular Forms**

Assuming the existence of the hypothetical motive $M(f)$ attached to a Siegel modular form f of degree g, H. Yoshida [161] proved an interesting fact that $M(f)$ has at most $g+1$ period invariants. I shall describe his results in some detail following his papers [160, 161, 163].

First of all we start with listing major historical events concerning critical values of zeta functions.

Around 1670, Gottfried W. Leibniz (1646-1716) found the following identity

$$\sum_{k=0}^{\infty} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

In 1735, Leonhard Euler (1707-1783) discovered the following interesting identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

experimentally and also in 1742 showed the following fact

$$\frac{\zeta(2n)}{\pi^{2n}} \in \mathbb{Q}, \quad n = 1, 2, 3, \cdots \in \mathbb{Z}^+.$$

In 1899, Adolf Hurwitz (1859-1919) showed the following fact

$$\sum_{z} z^{-4n}/\varpi^{4n} \in \mathbb{Q}, \quad n = 1, 2, 3, \cdots \in \mathbb{Z}^+,$$

where z extends over all nonzero Gaussian integers and

$$\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

In 1959, Goro Shimura (1930-) proved that

$$\frac{L(n,\Delta)}{(2\pi i)^n c^{\pm}(\Delta)} \in \mathbb{Q}, \qquad 1 \leqq n \leqq 11, \quad \pm 1 = (-1)^n,$$

where

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n)\, q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \qquad q = \exp(2\pi i \tau)$$

is the cusp form of weight 12 with respect to $SL(2,\mathbb{Z})$ and $c^{\pm}(\Delta) \in \mathbb{R}^{\times}$. Here

$$L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

is the L-function of $\Delta(\tau)$ and $\tau(n)$ is the so-called Ramanujan tau function which has the following property

$$|\tau(p)| \leq 2\, p^{11/2} \quad \text{for all primes } p.$$

The above property was proved by Pierre Deligne (1944-) in 1974. For instance, $\tau(2) = -24,\ \tau(3) = 252,\ \tau(5) = 4830,\ \tau(7) = -16744,\ \tau(11) = 534612, \tau(13) = -577738$.

In 1977, Shimura [130] proved in a similar way that for a Hecke eigenform $f \in S_k(\Gamma_0(N), \psi)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\left(\frac{L(n,f)}{(2\pi i)^n c^{\pm}(f)} \right)^{\sigma} = \frac{L(n, f^{\sigma})}{(2\pi i)^n c^{\pm}(f^{\sigma})}, \quad 1 \leqq n \leqq k-1, \quad \pm 1 = (-1)^n,$$

where $c^{\pm}(f^{\sigma}) \in \mathbb{C}^{\times}$. By these results, it was expected that the critical values of zeta functions are related to periods of integrals. Here the notion of critical values, which is generally accepted now, can be defined as follows. Suppose that a zeta function $Z(s)$ multiplied by its gamma factor $G(s)$ satisfies a functional equation of standard type under the symmetry $s \longrightarrow v - s$. Then $Z(n)$, $n \in \mathbb{Z}$ is a critical value of $Z(s)$ if both of $G(n)$ and $G(v-n)$ are finite.

In 1979, Pierre Deligne [24] published a general conjecture which gives a prediction on critical values of the L-function of a motive. For a nice concise exposition of the theory of motives, we refer the reader to a paper of Jannsen [69]. For more comprehensive information, we refer to [70].

Let E be an algebraic number field with finite degree $l = [E : \mathbb{Q}]$. Let J_E be the set of all isomorphisms of E into $\mathbb{C}$. We put $R = E \otimes_{\mathbb{Q}} \mathbb{C}$. Let M be a motive over $\mathbb{Q}$ with coefficients in E. Roughly speaking motives arise as direct summands of the cohomology of a smooth projective algebraic variety defined over $\mathbb{Q}$. Naively they may be defined by a collection of realizations satisfying certain axioms. A motive M has at least three realizations : the Betti realization, the de Rham realization and the λ-adic realization.

First we let $H_B(M)$ be the Betti realization of M. Then $H_B(M)$ is a free module over E of rank $d := d(M)$. We put $H_B(M)_{\mathbb{C}} := H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}$. We have the involution F_{∞} acting on $H_B(M)_{\mathbb{C}}$ E-linearly. Therefore we obtain the the eigenspace decomposition

$$H_B(M)_{\mathbb{C}} = H_B^{+}(M) \oplus H_B^{-}(M), \tag{12.1}$$

where $H_B^{+}(M)$ (resp. $H_B^{-}(M)$) denotes the $(+1)$-eigenspace (resp. the (-1)-eigenspace) of $H_B(M)$. We let d^{+} (resp. d^{-}) be the dimension $H_B^{+}(M)$ (resp. $H_B^{-}(M)$). Furthermore $H_B(M)_{\mathbb{C}}$ has the Hodge decomposition into $\mathbb{C}$-vector spaces :

$$H_B(M)_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}(M), \tag{12.2}$$

where $H^{p,q}(M)$ is a free R-module. A motive M is said to be *of pure weight* $w := w(M)$ if $H^{p,q}(M) = \{0\}$ whenever $p + q \neq w$. From now on we shall assume that M is of pure weight.

Secondly we let $H_{\rm DR}(M)$ be the de Rham realization of M that is a free module over E of rank d. Let

$$H_{\rm DR}(M) = F^{i_1} \supsetneqq F^{i_2} \supsetneqq \cdots \supsetneqq F^{i_m} \supsetneqq F^{i_{m+1}} = \{0\} \tag{12.3}$$

be a decreasing Hodge filtration so that there are no different filtrations between successive members. The choice of members i_ν may not be unique for F^{i_ν}. For the sake of simplicity, we assume that i_ν is chosen for $1 \leq \nu \leq m$ so that it is the maxium number. We put

$$s_\nu = \text{rank } H^{i_\nu, w-i_\nu}(M), \qquad 1 \leq \nu \leq m,$$

where rank means the rank as a free R-module. Let

$$I : H_B(M)_{\mathbb{C}} \longrightarrow H_{\rm DR}(M)_{\mathbb{C}} = H_{\rm DR}(M) \otimes_E \mathbb{C}$$

be the comparison isomorphism which satisfies the conditions

$$I\left(\bigoplus_{p' \geq p} H^{p',q}(M)\right) = F^p \otimes_{\mathbb{Q}} \mathbb{C}. \tag{12.4}$$

According to (12.4), we get

$$s_\nu = \dim_E F^{i_\nu} - \dim_E F^{i_{\nu+1}}, \quad \dim_E F^{i_\nu} = s_\nu + s_{\nu+1} + \cdots + s_m, \quad 1 \leq \nu \leq m.$$

We choose a basis $\{w_1, \cdots, w_d\}$ of $H_{\rm DR}(M)$ over E so that $\{w_{s_1+s_2+\cdots+s_{\nu-1}+1}, \cdots, w_d\}$ is a basis of F^{i_ν} for $1 \leq \nu \leq m$. We observe that

$$d = s_1 + s_2 + \cdots + s_m \quad \text{all } s_\nu > 0 \text{ with } 1 \leq \nu \leq m. \tag{12.5}$$

We are in a position to describe the fundamental periods of M that Yoshida introduced. Let $\{v_1^+, v_2^+, \cdots, v_{d^+}^+\}$ (resp. $\{v_1^-, v_2^-, \cdots, v_{d^-}^-\}$) be a basis of $H_B^+(M)$ (resp. $H_B^-(M)$) over E. Writing

$$I(v_j^{\pm}) = \sum_{i=1}^d x_{ij}^{\pm} w_i, \quad x_{ij}^{\pm} \in R, \quad 1 \leq j \leq d^{\pm}, \tag{12.6}$$

we obtain a matrix $X^+ = (x_{ij}^+) \in R^{(d,d^+)}$ and a matrix $X^- = (x_{ij}^-) \in R^{(d,d^-)}$. We recall that $R^{(m,n)}$ denotes the set of all $m \times n$ matrices with entries in R. Let P_M be the lower parabolic subgroup of $GL(d)$ which corresponds to the partition (12.5). Let $P_M(E)$ be the group of E-rational points of P_M. Then the coset of X^+ (resp. X^-) in

$$P_M(E)\backslash R^{(d,d^+)}/GL(d^+, E) \quad \big(\text{resp. } P_M(E)\backslash R^{(d,d^-)}/GL(d^-, E)\big)$$

is independent of the choice of a basis. We set $X_M = (X^+, X^-) \in R^{(d,d)}$. Then it is easily seen that the coset of X_M in

$$P_M(E)\backslash R^{(d,d)}/\big(GL(d^+, E) \times GL(d^-, E)\big)$$

is independent of the choice of a basis, i.e., well defined. A $d \times d$ matrix $X_M = (X^+, X^-)$ is called a *period matrix* of M.

For an m-tuple $(a_1, \cdots, a_m) \in \mathbb{Z}^m$ of integers, we define a character λ_1 of P_M by

$$\lambda_1\left(\begin{pmatrix} P_1 & 0 & \dots & 0 \\ * & P_2 & \dots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & P_m \end{pmatrix}\right) = \prod_{j=1}^{m} \det(P_j)^{a_j}, \quad P_j \in GL(s_j), \quad 1 \le j \le m.$$

For a pair (k^+, k^-) of integers, we define a character λ_2 of $GL(d^+) \times GL(d^-)$ by

$$\lambda_2\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = (\det A)^{k^+} (\det B)^{k^-}, \qquad A \in GL(d^+),\ B \in GL(d^-).$$

A polynomial f on $R^{(d,d)}$ rational over $\mathbb{Q}$ is said to be *of the type* $\{(a_1, \cdots, a_m); (k^+, k^-)\}$ or *of the type* (λ_1, λ_2) if f satisfies the following condition

$$\text{(12.7)} \quad f(pxq) = \lambda_1(p)\lambda_2(q)f(x) \qquad \text{for all } \ p \in P_M,\ q \in GL(d^+) \times GL(d^-).$$

We now assume that f is a nonzero polynomial on $R^{(d,d)}$ of the type $\{(a_1, \cdots, a_m); (k^+, k^-)\}$. Let $X_M = (X^+, X^-)$ be a period matrix of a motive M as before. Then it is clear that $f(X_M)$ is uniquely determined up to multiplication by elements in $E^\times$. We call $f(X_M)$ a *period invariant* of M of the type $\{(a_1, \cdots, a_m); (k^+, k^-)\}$. Hereafter we understand the equality between period invariants mod $E^\times$.

We now consider the following special polynomials of the type (λ_1, λ_2) :

I. Let $f(x) = \det(x)$ for $x \in R^{(d,d)}$.

It is easily seen that $f(x)$ is of the type $\{(1, 1, \cdots, 1); (1, 1)\}$. Then $f(X_M)$ is nothing but Deligne's period $\delta(M)$.

II. Let $f^+(x)$ be the determinant of the upper left $d^+ \times d^+$-submatrix of $x \in R^{(d,d)}$. It is easily checked that $f^+(x)$ is of the type

$$\{(\overbrace{1, 1, \dots, 1}^{p^+}, 0, \dots, 0); (1, 0)\},$$

where p^+ is a positive integer such that $s_1 + s_2 + \cdots + s_{p^+} = d^+$. We note that $f^+(X_M)$ is Deligne's period $c^+(M)$.

III. Let $f^-(x)$ be the determinant of the upper right $d^- \times d^-$-submatrix of x. Then $f^-(x)$ is of the type

$$\{(\overbrace{1, 1, \dots, 1}^{p^-}, 0, \dots, 0); (0, 1)\}$$

and $f^-(X_M)$ is Deligne's period $c^-(M)$. Here p^- is a positive integer such that $s_1 + s_2 + \cdots + s_{p^-} = d^-$.

Either one of the above conditions is equivalent to that $F^\mp(M)$, hence also $c^\pm(M)$ can be defined (cf. [23], §1, [160], §2). We have $F^\mp(M) = F^{i_{p^\pm+1}}(M)$; $F^\pm(M)$ can be defined if M has a critical value. Let $\mathcal{P} = \mathcal{P}(M)$ denote the set of integers p such that $s_1 + s_2 + \cdots + s_p < \min(d^+, d^-)$. Yoshida (cf. [161], Theorem 3) showed that for every $p \in \mathcal{P}$, there exists a non-zero polynomial f_p of the type

$$\big\{(\overbrace{2,\ldots,2}^{p},\overbrace{1,\ldots,1}^{m-2p},\overbrace{0,\ldots,0}^{p});(1,1)\big\}$$

and that every polynomial satisfying (12.7) can be written uniquely as a monomial of $\det(x)$, $f^+(x)$, $f^-(x)$, $f_p(x)$, $p \in \mathcal{P}$. We put $c_p(M) = f_p(X_M)$. We call $\delta(M)$, $c^{\pm}(M)$, $c_p(M)$, $p \in \mathcal{P}$ the *fundamental periods* of M. Therefore any period invariant of M can be written as a monomial of the fundamental periods. Moreover Yoshida showed that if a motive M is constructed from motives $M_1, \cdots, M_t$ of pure weight by standard algebraic operations then the fundamental periods of M can be written as monomials of the fundamental periods of $M_1, \cdots, M_t$. He proved that a motive M has at most $\min(d^+, d^-) + 2$ fundamental periods including Deligne's periods $\delta(M)$ and $c^{\pm}(M)$.

Thirdly we let $H_\lambda(M)$ be the λ-adic realization of M. We note that $H_\lambda(M)$ is a free module over E_λ of rank d. We have a continuous λ-adic representation of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}\big(\overline{\mathbb{Q}}/\mathbb{Q}\big)$ on $H_\lambda(M)$ for each prime λ. Also there is an isomorphism $I_\lambda : H_B(M) \otimes_E E_\lambda \longrightarrow H_\lambda(M)$ which transforms the involution F_∞ into the complex conjugation.

We recall that an integer $s = n$ is said to be *critical* for a motive M if both the infinite Euler factors $L_\infty(M, s)$ and $L_\infty(\check{M}, s)$ are holomorphic at $s = n$. Here $L(M, s)$ denotes the complex L-function attached to M and $\check{M}$ denotes the dual motive of M. Such values $L(M, n)$ are called *critical values* of $L(M, s)$. Deligne proposed the following.

Conjecture (Deligne [23]**).** *Let M be a motive of pure weight and $L(M, s)$ the L-function of M. Then for critical values $L(M, n)$, one has*

$$\frac{L(M,n)}{(2\pi i)^{d^{\pm}}\, c^{\pm}(M)} \in E, \quad d^{\pm} := d^{\pm}(M),\ \pm 1 = (-1)^n.$$

Indeed Deligne showed that $c^{\pm}(M) \in R^{\times}$ and Yoshida showed that other period invariants are elements of $R^{\times}$.

Remark 12.1. The Hodge decomposition (12.2) determines the gamma factors of the conjectural functional equation of $L(M, s)$. Conversely the gamma factor of the functional equation of $L(M, s)$ determines the Hodge decomposition if M is of pure weight.

Let $f \in S_k(\Gamma_g)$ be a nonzero Hecke eigenform on $\mathbb{H}_g$. Let $L_{\mathrm{st}}(s, f)$ and $L_{\mathrm{sp}}(s, f)$ be the standard zeta function and the spinor zeta function of f respectively. For the sake of simplicity we use the notations $L_{\mathrm{st}}(s, f)$ and $L_{\mathrm{sp}}(s, f)$ instead of $D_f(s)$ and $Z_f(s)$ (cf. §8) in this section. We put $w = kg - \frac{1}{2}g(g+1)$. We have a normalized Petersson inner product $\langle\ ,\ \rangle$ on $S_k(\Gamma_g)$ given by

$$\langle F, F\rangle = \mathrm{vol}\big(\Gamma_g\backslash\mathbb{H}_g\big)^{-1}\int_{\Gamma_g\backslash\mathbb{H}_g} |f(\Omega)|^2 \big(\det Y\big)^{k-g-1}[dX][dY], \quad F \in S_k(\Gamma_g),$$

where $\Omega = X + iY \in \mathbb{H}_g$ with real $X = (x_{\mu\nu}),\ Y = (y_{\mu\nu})$, $[dX] = \bigwedge_{\mu\le\nu} dx_{\mu\nu}$ and $[dY] = \bigwedge_{\mu\le\nu} dy_{\mu\nu}$.

We assume the following (A1)-(A6) :

(A1) The Fourier coefficients of f are contained in a totally real algebraic number field E.
(A2) There exist motives $M_{\rm st}(f)$ and $M_{\rm sp}(f)$ over $\mathbb{Q}$ with coefficients in E satisfying the conditions

$$L\big(M_{\rm st}(f),s\big)=\big(L_{\rm st}(s,f^\sigma)\big)_{\sigma\in J_E}\quad\text{and}\quad L\big(M_{\rm sp}(f),s\big)=\big(L_{\rm sp}(s,f^\sigma)\big)_{\sigma\in J_E}.$$

(A3) Both $M_{\rm st}(f)$ and $M_{\rm sp}(f)$ are of pure weight.
(A4) We assume

$$\bigwedge^{2g+1} M_{\rm st}(f)\cong T(0),$$

$$\begin{aligned}H_B(M_{\rm st}(f))\otimes_{\mathbb{Q}}\mathbb{C}= &\quad H^{0,0}(M_{\rm st}(f))\\ \bigoplus_{i=1}^{g} &\quad \Big(H^{-k+i,k-i}(M_{\rm st}(f))\oplus H^{k-i,-k+i}(M_{\rm st}(f))\Big).\end{aligned}$$

We also assume that the involution F_∞ acts on $H^{0,0}(M_{\rm st}(f))$ by $(-1)^g$.
(A5) We assume

$$\bigwedge^{2^g} M_{\rm sp}(f)\cong T(2^{g-1}w),$$

$$H_B(M_{\rm sp}(f))\otimes_{\mathbb{Q}}\mathbb{C}=\bigoplus_{p,q}H^{p,q}(M_{\rm sp}(f)),$$

$$p=(k-i_1)+(k-i_2)+\cdots+(k-i_r),\ q=(k-j_1)+(k-j_2)+\cdots+(k-j_s),$$

$$r+s=g,\qquad 1\le i_1<\cdots<i_r\le g,\qquad 1\le j_1<\cdots<j_s\le g,$$

$$\{i_1,\cdots,i_r\}\cup\{j_1,\cdots,j_s\}=\{1,2,\ldots,g\},$$

including the cases $r=0$ or $s=0$.
(A6) If $w=kg-\frac{1}{2}g(g+1)$ is even, then the eigenvalues $+1$ and -1 of F_∞ on $H^{p,p}(M_{\rm sp}(f))$ occur with the equal multiplicities.

Let $J_E=\{\sigma_1,\sigma_2,\ldots,\sigma_l\}$, $l=[E:\mathbb{Q}]$ and write $x\in R\cong\mathbb{C}^{J_E}$ as $x=(x^{(1)},x^{(2)},\cdots,x^{(l)})$, $x^{(i)}\in\mathbb{C}$ so that $x^{(i)}=x^{\sigma_i}$ for $x\in E$. Yoshida showed that when $k>2g$, assuming Deligne's conjecture, one has

$$c^{\pm}(M_{\rm st}(f))=\pi^{kg}\big(\langle f^\sigma,f^\sigma\rangle\big)_{\sigma\in J_E}.$$

He proved the following interesting result (cf. Yoshida [161], Theorem 14).

Theorem 12.1. *Let the notation be the same as above. We assume that two motives over $\mathbb{Q}$ having the same L-function are isomorphic (Tate's conjecture). Then there exist $p_1,\ p_2,\ \cdots,\ p_r\in\mathbb{C}^\times$, $1\le r\le g+1$ such that for any fundamental period $c\in R^\times$ of $M_{st}(f)$ or $M_{sp}(f)$, we have*

$$c^{(1)}=\alpha\,\pi^A\,p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$$

with $\alpha\in\overline{\mathbb{Q}}^\times$ and non-negative integers A, a_i, $1\le i\le r$.

Remark 12.2. It is widely believed that the zeta function of the Siegel modular variety $\mathcal{A}_g:=\Gamma_g\backslash\mathbb{H}_g$ can be expressed using the spinor zeta functions of (not necessarily holomorphic) Siegel modular forms:

$$\zeta\big(s,\mathcal{A}_g\big)\doteqdot\prod_f L_{\rm sp}(s,f).$$

Yoshida proposed the following conjecture.

Conjecture (Yoshida [161]**).** *If one of two motives $M_{st}(f)$ and $M_{sp}(f)$ is not of pure weight, then the associated automorphic representation to f is not tempered. Furthermore f can be obtained as a lifting from lower degree forms.*

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